FUNCTIONS OF A SPACETIME VARIABLE

Mathematics and Computer Education, 36(2002), pp. 231-239

Nicolae A. Borota, *Office of Assessment*

New Jersey Department of Education

Trenton, NJ 08625-0500

Thomas J. Osler, *Department of Mathematics*

Rowan University

Glassboro, NJ 08028

1. INTRODUCTION

Like the familiar complex numbers $x+iy$, spacetime numbers $x+jt$ are a two dimensional extension of the one-dimensional real numbers *x*. In our previous paper [2], we explained how to add, subtract, multiply and divide spacetime numbers. We saw that some features of this arithmetic are identical to arithmetic with complex numbers (like addition and subtraction), but other manipulations (like multiplication and division) were very different. In these papers we attempt to introduce a subject of importance in applications of mathematics that has previously only been presented at a level appropriate for researchers or graduate students.

We begin with a brief review of key concepts from [2], then introduce the spacetime version of Euler's Formula and use it to represent spacetime numbers. We end by showing how to extend any familiar function of the real variable *x* to the spacetime variable $x+jt$. We avoid a rigorous definition-lemma-theorem style in favor of a playful, self-discovery approach. Rigorous presentations are easily found in the references. This paper can be presented to students in calculus, real analysis or complex analysis. The minimum requirement is a familiarity with complex numbers, complex arithmetic, series and Euler's formula for the complex exponential.

Problems are given throughout the paper with selected answers appearing in the final section.

2. REVIEW OF IMPORTANT CONCEPTS

While it would be best for the reader to study our first paper [2] before continuing, we include in this section all the relevant review material.

Complex arithmetic is based upon the simple notion that $i^2 = -1$ while spacetime arithmetic is based upon $j^2 = 1$. This somewhat docile equation leads to many important results, especially in the area of relativity theory [4]. We write spacetime numbers in the form $z = x + jt$. We call j the spacetime unit, x the real part, and t the time part of our spacetime number.

In motivating complex numbers for beginners, we first note that the equation $i^2 = -1$ has no real solutions. Thus we consider i to be a new type of number. *It is important to note that while the equation $j^2 = 1$ does have the real solutions $j = \pm 1$, we still think of j as a new type of number. We do not replace it by 1 or –1.*

There are important similarities between the complex and the spacetime number systems, which are summarized, in the following tables:

Powers of i:	**Powers of j:**

$i^0 = 1$ $i^1 = i$ $i^2 = -1$ $i^3 = -i$ $i^4 = 1$	$j^0 = 1$ $j^1 = j$ $j^2 = 1$ $j^3 = j$ $j^4 = 1$

Table 1: Powers of i and j

	Complex Arithmetic	**Spacetime Arithmetic**
Numbers	$Z = X + iY, z = x + iy$	$Z = X + jT, z = x + jt$
Addition	$Z + z = (X + x) + i(Y + y)$	$Z + z = (X + x) + j(T + t)$
Subtraction	$Z - z = (X - x) + i(Y - y)$	$Z - z = (X - x) + j(T - t)$
Multiplication	$Zz = (Xx - Yy) + i(Yx + yX)$	$Zz = (Xx + Tt) + j(Tx + tX)$
Division	$\frac{Z}{z} = \frac{X + iY}{x + iy} \times \frac{x - iy}{x - iy}$ $= \frac{(Xx + Yy) + i(Yx - Xy)}{(x^2 + y^2)}$	$\frac{Z}{z} = \frac{X + jT}{x + jt} \times \frac{x - jt}{x - jt}$ $= \frac{(Xx - Tt) + j(Tx - Xt)}{(x^2 - t^2)}$

Table 2: Complex Arithmetic vs. Spacetime Arithmetic

The division of two spacetime numbers leads to a difficulty when we divide by $z = x + jt$ where $x^2 - t^2 = 0$. (See Table 2.) This "difficulty" leads to three distinct forms of spacetime numbers. Given an arbitrary spacetime number of the form $z = x + jt$, it is categorized into one of the following three types:

Type I: $x^2 = t^2$ Light-like

Type II: $x^2 > t^2$ Space-like

Type III: $x^2 < t^2$ Time-like

We can represent the spacetime number $x + jt$ as a point (x,t) in the so called *spacetime plane,* in exactly the same way that the complex number $x + iy$ is represented by the point (x, y) in the complex plane. The following figure is a graphical representation which shows the regions in which the three types of spacetime numbers reside:

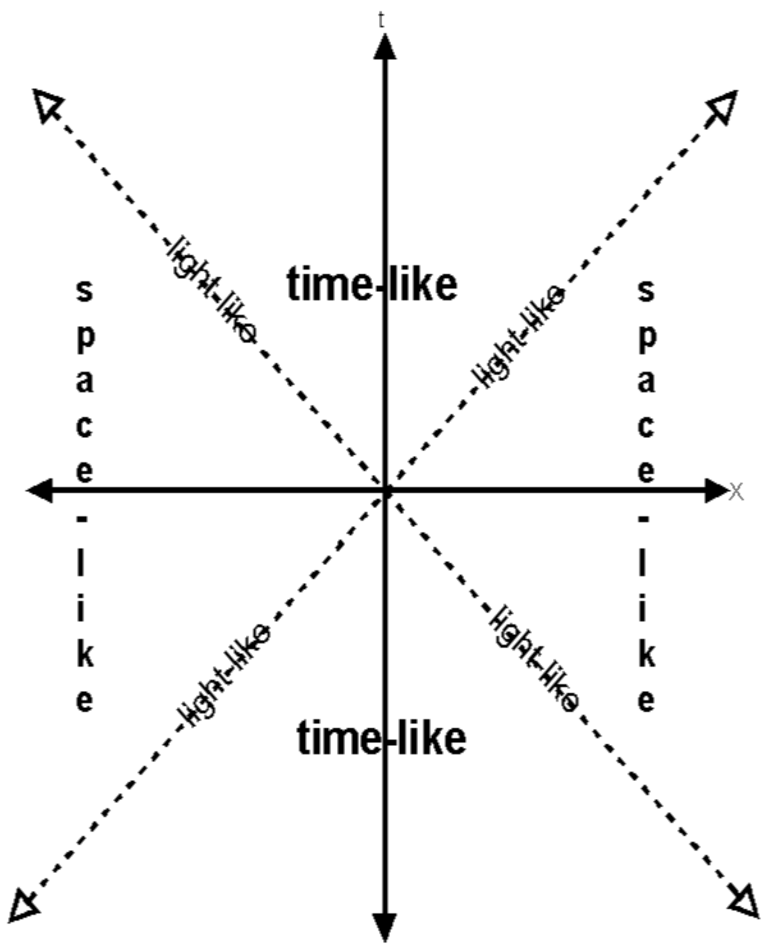

Figure 1: Three types of spacetime numbers

Problems

2.1 Let $z = 2 + 3j$ and $w = 3 - 2j$. Find (a) $z + w$, (b) $z - w$, (c) zw, and (d) z/w.

2.2 Let $z = 3 + 2j$. Find (a) z^2, and (b) z^3.

2.3 Divide $2 + 3j$ by $1 + j$.

2.4 Classify each of the following as (A) space-like, (B) time-like, or (C) light-like.

(a) 3 + 2*j*, (b) -3 + 2*j,* (c) 2 + 2*j*, (d) -1 +6*j*, (e) -1 - *j*, (f) -3 -4*j*.

3. THE SPACETIME EXPONENTIAL

Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$, is one of the most important results in mathematical analysis. It allows us to write a complex number of the form $x + iy$ in terms of the exponential function, using polar coordinates, as $x + iy = re^{i\theta}$. Since we have seen similarities between the complex number system and the spacetime number system, it makes sense to try to find a spacetime version of Euler's Formula. In the following we manipulate expressions *formally* rather than *rigorously*.

We begin with the power series expansion of e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

If we set $x = j\alpha$ and substitute into the above power series, we obtain

$$e^{j\alpha} = 1 + \frac{j\alpha}{1!} + \frac{(j\alpha)^2}{2!} + \frac{(j\alpha)^3}{3!} + \frac{(j\alpha)^4}{4!} + \cdots$$

$$= 1 + \frac{j\alpha}{1!} + \frac{\alpha^2}{2!} + \frac{j\alpha^3}{3!} + \frac{\alpha^4}{4!} + \cdots$$

$$= (1 + \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \cdots) + j(\frac{\alpha}{1!} + \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \cdots).$$

The first series in this last expression is $\cosh\alpha$ and the second is $\sinh\alpha$. This yields the spacetime version of Euler's formula:

(3.1) $$e^{j\alpha} = \cosh\alpha + j\sinh\alpha.$$

Because hyperbolic functions have replaced the corresponding trigonometric functions in Euler's formula, some authors refer to our *spacetime numbers* by the name *hyperbolic numbers.*

Problems

3.1 Show that $\sinh\alpha = \dfrac{e^{j\alpha} - e^{-j\alpha}}{2j}$.

3.2 Show that $\cosh\alpha = \dfrac{e^{j\alpha} + e^{-j\alpha}}{2}$.

3.3 Using the series $\sin\theta = \sum_{n=0}^{\infty} \dfrac{(-1)^n \theta^{2n+1}}{(2n+1)!}$, show that $\sin j\theta = j\sin\theta$.

3.4 Using the series $\cos\theta = \sum_{n=0}^{\infty} \dfrac{(-1)^n \theta^{2n}}{(2n)!}$, show that $\cos j\theta = \cos\theta$.

3.5 Using $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$, show that.

$$\sin(x + jt) = \sin x \cos t + j\sin t \cos x$$

3.6 Show that $\cos(x + jt) = \cos x \cos t + j\sin x \sin t$.

4. EXPONENTIAL FORM OF A SPACETIME NUMBER

Any complex number $z = x + iy$ can also be written in the polar form $z = re^{i\theta}$. Here we have the transformation equations

(4.1) $$x = r\cos\theta,\ y = r\sin\theta, \text{ and}$$

(4.2) $$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\frac{y}{x}.$$

Is it possible to perform a similar transformation with spacetime numbers? Can we find real numbers ρ and α so that $x + jt = \rho e^{j\alpha}$? Let us see.

Suppose $z = x + jt$ is a space-like number, so that $x^2 > t^2$. Then we can write

$$x + jt = \rho e^{j\alpha}, \text{ and}$$

$$x - jt = \rho e^{-j\alpha}.$$

Multiplying these last two expressions we get

$$x^2 - t^2 = \rho^2, \text{ and so}$$

(4.3) $$\rho = \pm\sqrt{x^2 - t^2}.$$

Also from (3.1) we see that $x + jt = \rho e^{j\alpha} = \rho(\cosh\alpha + j\sinh\alpha)$, so we have

(4.4) $$x = \rho\cosh\alpha, \text{ and}$$

$$t = \rho\sinh\alpha.$$

From (4.4) we see that the + sign is used in (4.3) if $x > 0$, otherwise the $-$ sign is used.

Dividing these last two expressions we get $\frac{t}{x} = \tanh\alpha$, and so

$$\alpha = \tanh^{-1}\frac{t}{x}.$$

We call ρ the *spacetime modulus* and α the *hyperbolic angle.*

In summary, we have shown that if $z = x + jt$ is a space-like number, so that $x^2 > t^2$, then we can write $x + jt = \rho e^{j\alpha}$ where

$$\rho = \pm\sqrt{x^2 - t^2} \quad \begin{pmatrix} Use + if\ x > 0, \\ use - if\ x < 0. \end{pmatrix}, \text{ and } \alpha = \tanh^{-1}\frac{t}{x}.$$

$$x = \rho\cosh\alpha \text{ and } t = \rho\sinh\alpha .$$

In a similar way we can show that if $z = x + jt$ is a time-like number, so that $x^2 < t^2$, then we can write $x + jt = j\rho e^{j\alpha}$, where

$$\rho = \pm\sqrt{t^2 - x^2} \begin{pmatrix} Use + if \;\; t > 0, \\ use - if \;\; t < 0. \end{pmatrix}, \text{ and } \alpha = \tanh^{-1}\frac{x}{t},$$

$$x = \rho\sinh\alpha \text{ and } t = \rho\cosh\alpha .$$

If $z = x + jt$ is a light-like number, $x^2 = t^2$, then no exponential representation is possible.

Problems

4.1 Express each of the following spacetime numbers in the form $\rho e^{j\alpha}$.

(a) 3 + 2*j*, (b) 3 – 2*j*, (c) –3 + 2*j*, (d) –3 – 2*j*.

4.2 Find ρ and α such that $2 + 3j = j\rho e^{j\alpha}$.

4.3 Find *x* and *t* such that $x + jt = 2e^{5j}$.

4.4 Show that $\rho e^{j\alpha}$, (where ρ and α are real numbers), is a space-like number.

4.5 Using the fact that any space-like number can be written in the form $\rho e^{j\alpha}$, prove that the product of two space-like numbers is always space-like.

5. THE GENERAL FORM OF SPACETIME FUNCTIONS:

In this section we will find a general result that allows us to find the spacetime version of any given real function. To understand our objective, consider the simple function $y = x^2$. If we replace *x* and *y* by spacetime variables $z = x + jt$ and $w = u + jv$ we get

$w = z^2$, so

$u + jv = (x + jt)^2 = (x^2 + t^2) + 2xtj$.

Therefore $u = x^2 + t^2$, and $v = 2xt$. How can we do this with any familiar real function? Given $y = f(x)$, how do we find space part $u(x,t)$ and time part $v(x,t)$ so that $u(x,t) + jv(x,t) = f(x + jt)$? This is our objective in this section.

Since $j^2 = 1$ it is natural to think of j as being $+1$ or -1. However, recall that the hyperbolic unit j is not supposed to be merely $+1$ or -1, but a new type of number. Nevertheless, as a tool for heuristic discovery, this substitution will prove to be of great use.

Substitution tool for discovery:

In any correct relation where the spacetime unit j appears, try replacing j by 1 *or* -1, *then explore its consequences.*

If we take $f(z) = f(x + jt) = u(x,t) + jv(x,t)$ and replace j by $+1$ and -1, we generate the following:

(5.1) For $j = 1$, $f(x + t) = u(x,t) + v(x,t)$.

(5.2) For $j = -1$, $f(x - t) = u(x,t) - v(x,t)$.

If we add (5.1) and (5.2), we have

(5.3) $$u(x,t) = \frac{f(x+t) + f(x-t)}{2} .$$

If we subtract (5.2) from (5.1), we have

(5.4) $$v(x,t) = \frac{f(x+t) - f(x-t)}{2} .$$

We can use (5.3) and (5.4) in a rigorous treatment of this subject to define the space part and time part of any familiar function.

Example

Let's examine $\cos z$. Using (5.3) to solve for the space part of $\cos z$,

$$u(x,t) = \frac{\cos(x+t)+\cos(x-t)}{2}$$

$$= \frac{\cos x\cos t+\sin x\sin t}{2} + \frac{\cos x\cos t-\sin x\sin t}{2}$$

$$= \cos x\cos t\,.$$

Using (5.4) to solve for the time part of $\cos z$,

$$v(x,t) = \frac{\cos(x+t)-\cos(x-t)}{2}$$

$$= \frac{\cos x\cos t-\sin x\sin t}{2} - \frac{\cos x\cos t+\sin x\sin t}{2}$$

$$= -\sin x\sin t\,.$$

Therefore, combining the space and time parts, we get the formula for $\cos z$

(5.5) $\cos z = \cos x\cos t - j\sin x\sin t$.

Problems

5.1. Find the space and time parts of the function $z^2 = (x+jt)^2$ in two ways, first using elementary algebra, and second using (5.3) and (5.4).

5.2. Find the space and time parts of the function $z^3 = (x+jt)^3$ in two ways, first using elementary algebra, and second using (5.3) and (5.4).

5.3 Find the space and time parts of the function $\sin(x+jt)$ using (5.3) and (5.4).

5.4 (a) Find $\tan(x+jt)$ using (5.3) and (5.4). (b) Show that $\tan(x+jt) = \dfrac{\sin(x+jt)}{\cos(x+jt)}$.

6. COMMENTS ON THE REFERENCES

We plan to submit another paper, which will show applications to special relativity in the same informal style as this one.

For further study we suggest the excellent paper by Garret Sobczyk [16]. We strongly recommend this paper even though it does use the terminology of modern abstract algebra. We also recommend the paper by Fjelstad [4]. In this paper Fjelstad explains how he and his students rediscovered spacetime numbers (he calls them perplex numbers). The references by Band [1], Majernik [9] and Ronveaux [11] are letters in response to the paper of Fjelstad and we think the reader will find these very interesting. Readers familiar with fractals and the Mandelbrot set will find the papers of Senn [12] and Metzier [10] interesting because they show what the Mandelbrot set looks in the spacetime plane. The paper by Lambert [7] is an amusing critique of this entire enterprise (in French). References [5] and [6] are graduate level textbooks. The remaining references are research papers in the theory and application of spacetime numbers and other related number systems.

Our list of references represents only a small part of the available literature. Note that any material on Clifford algebras is of potential interest since spacetime numbers are a special case of these.

7. ANSWERS TO PROBLEMS

2.1 (a) $5+j$, (b) $-1+5j$, (c) $5j$, (d) $\frac{12}{5}+\frac{13}{5}j$.

2.2 (a) $13+12j$, (b) $63+62j$.

2.3 Division by light-like numbers is not permissible.

2.4 (a) space-like, (b) space-like, (c) light-like, (d) time-like, (e) light-like, (f) time-like.

4.1 (a) $\sqrt{5}\exp\left(\tanh^{-1}\frac{2}{3}\right)$, (b) $\sqrt{5}\exp\left(-\tanh^{-1}\frac{2}{3}\right)$, (c) $-\sqrt{5}\exp\left(-\tanh^{-1}\frac{2}{3}\right)$,

(d) $-\sqrt{5}\exp\left(\tanh^{-1}\frac{2}{3}\right)$.

4.2 $\rho = \sqrt{5}$ $\quad \alpha = \tanh^{-1}\frac{3}{2}$

4.3 $x = 2\cosh\alpha$ $\quad y = 2\sinh\alpha$

5.1 space part: $x^2 + t^2$ time part: $2xt$.

5.2 space part: $x^3 + 3xt$ time part: $3x^2 t + t^3$.

5.3 space part: $\sin x \cos t$ time part: $\sin t \cos x$.

5.4 $\frac{\sin x \cos t + j \sin t \cos x}{\cos^2 x \cos^2 t - \sin^2 x \sin^2 t}$.

REFERENCES

[1] Band, W., *Comments on "Extending relativity via the perplex numbers"[Am. J. Phys. 54, 416(1986)],* Am. J. Phys., 56(1988), p. 469.

[2] N. Borota, E. Flores, T. Osler, "Spacetime Numbers The Easy Way"*, Mathematics and Computer Education,* Vol. 34, No. 2, pp.159-168 (2000).

[3] W. K. Clifford, *Applications of Grassman's extensive algebra,* American Journal of Mathematics, 1 (1878), pp. 350-358.

[4] P. Fjelstad, *Extending relativity via the perplex numbers,* American Journal of Physics, 54:5 (1986), pp. 416-422.

[5] D. Hestenes, *Space Time Algebra,* Gordon and Breach, 1966.

[6] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics,* Kluwer, Dordrecht, 1992.

[7] Lambert,-Dominique*, Les nombres complexes hyperboliques: des complexes qui*

nous laissent perplexes. [Hyperbolic complex numbers: complexes that leave us perplexed]
Rev.-Questions-Sci. [Revue-des-Questions-Scientifiques] 166 (1995), no. 4, 383--400.

[8] Majernik V., *Basic space-time transformations expressed by means of two component number systems,* Acta Phys. Polon. A, 86(1994), no. 5, pp. 291-295.

[9] Majernik V., *The perplex numbers are in fact the binary numbers,* Am. J. Phys., 56(1988), no. 8, p. 763.

[10] Metzier, Wolfgang, *The "mystery" of the quadratic Mandelbrot set,* Am. J. Phys., 62(1994), pp. 813-814.

[11] Ronveaux, Andre, *About "perplex numbers",* Am. J. Phys., 55(1987), p. 392.

[12] Senn, Peter, *The Mandelbrot set for binary numbers,* Am. J. Phys., 58(1990), p. 1018.

[13] G. Sobczyk, *Spacetime vector analysis,* Physics Letters, 84A (1981), P. 45.

[14] Miller,-G.-R., Thaheem,-A.-B., *Derivatives of matrix order*, Arab-J.-Math.-Sci. [Arab-Journal-of-Mathematical-Sciences] 3 (1997), no. 1, 21--36.

[15] Ren,-Hungshan, *The hyperbolic quasifield of numbers on the plane and analytic functions on it*. I, Heilongjiang-Daxue-Ziran-Kexue-Xuebao [Natural-Science-Journal-of-Heilongjiang-University] 1988, no. 2, 9--13.

[16] G. Sobczyk, *The hyperbolic number plane,* The College Mathematics Journal, 26 (1995), pp.268-280.