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SPACETIME NUMBERS THE EASY WAY

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1. Introduction

The subject of spacetime numbers, better known as hyperbolic numbers, is not new, but is not well known because it is not discussed in standard textbooks. It is the purpose of this paper to demonstrate that the arithmetic of spacetime numbers can be introduced to precalculus students. All that the students need is a familiarity with the arithmetic of complex numbers.

Because knowledge of spacetime numbers is not widely known, researchers occasionally rediscover the subject (this is true of one of the authors). Spacetime numbers are a special case of the more general Clifford Algebras. Many books and articles on this subject begin by explaining that more students of mathematics and physics would profit from a familiarity with spacetime numbers, and then proceed to introduce the subject in a rigorous mathematical format. These rigorous presentations, full of mathematical jargon, present an unnecessary barrier to readers not so mathematically sophisticated. Students soon tire of the unfriendly new vocabulary placed before them by mathematical researchers and, consequently, continue to use traditional mathematical tools like complex numbers instead. This is unfortunate, because many of the features of these numbers can be understood by precalculus students. Spacetime numbers can be used with profit to solve problems in physics, especially in relativity theory. They also provide a nifty example for students in a first course in modern abstract algebra.

It is our purpose here to present the arithmetic of spacetime numbers in a gentle and fun manner. We will avoid a rigorous definition, lemma, and theorem approach in favor of a much lighter, intuitive, self-discovery presentation. We invite the reader to join us on a pleasant exploration of a new and useful idea, and leave it to the references to fill in the needed mathematical rigor. Spacetime numbers behave in many ways like the familiar complex numbers, and we will use this analogy to help us understand them. On the other hand, spacetime numbers at times have surprising features probably never seen before by most students. These spicy new ideas should keep our adventure lively. So let us begin. First we will review how complex numbers are often introduced to precalculus students.

2. Complex numbers, spacetime numbers and arithmetic.

In high school algebra courses, the complex numbers are often introduced in a simple natural way. The easy steps are:

(a) Argue that no number familiar to the students satisfies $i = \sqrt{-1}$.

(b) Argue that $i^0 = 1,\ i^1 = i,\ i^2 = -1, i^3 = -i, i^4 = 1, \cdots$

(c) Introduce the quantity $x + iy$, where x and y are real numbers, as a new kind of number. Call it a complex number.

(d) The arithmetic of these new numbers is invented using a simple principle. Manipulate using all the familiar rules of algebra, treating i as you would any algebraic variable with the exception that whenever the quantity i^2 occurs, we replace it with -1.

Using these four easy steps, the arithmetic of the complex number system becomes assessable to precalculus students. In a similar way, we can introduce the less familiar spacetime numbers.

(a') We begin with the surprising simple relation $j^2 = 1$, which has the two solutions $j = \pm 1$. Unlike (a) above, these are not new numbers. Nevertheless, we will continue using j as though it were new and not replacing it by the two numbers ± 1.

(b') The list of powers of j is even simpler than the list of powers of i: $j^0 = 1, j^1 = j, j^2 = 1, j^3 = j, \cdots$. The j's form a two cycle, while the powers of i form a four cycle.

(c') Introduce the quantity $x + jt$, (where both x and t are real) and call it a spacetime number. Here the number x is called the space part and t is the time part. These names are motivated by applications.

(d') We can discover how to add, subtract, multiply and divide spacetime numbers by using the familiar rules of algebra and replacing j^2 by 1. The results are summarized in the following table:

	Complex Arithmetic	**Spacetime Arithmetic**

Numbers	$Z = X + iY$ and $z = x + iy$	$Z = X + jT$ and $z = x + jt$
Addition	$Z + z = (X + x) + i(Y + y)$	$Z + z = (X + x) + j(T + t)$
Subtraction	$Z - z = (X - x) + i(Y - y)$	$Z - z = (X - x) + j(T - t)$
Multiplication	$Zz = (Xx - Yy) + i(Yx + yX)$	$Zz = (Xx + Tt) + j(Tx + tX)$
Division	$\frac{Z}{z} = \frac{X + iY}{x + iy} \times \frac{x - iy}{x - iy}$ $= \frac{(Xx + Yy) + i(Yx - Xy)}{(x^2 + y^2)}$	$\frac{Z}{z} = \frac{X + jT}{x + jt} \times \frac{x - jt}{x - jt}$ $= \frac{(Xx - Tt) + j(Tx - Xt)}{(x^2 - t^2)}$

Table 1: Summary of Complex and Spacetime Arithmetic

Notice that complex addition and subtraction is identical to spacetime addition and subtraction, but multiplication and division are different. The following are some problems for the reader. The answers are given at the end of the paper.

Problems:

2.1 Given two spacetime numbers, $Z = 3 + 7j$ and $z = -5 + 3j$. Calculate a) $Z + z$, b) $Z - z$, c) Zz, d) $\frac{Z}{z}$.

2.2 What would happen if we were

given $Z = 5 - j7 \; and \; z = 3 - j3$ and asked to calculate $\frac{Z}{z}$?

2.3 Verify the entries in Table 1 for spacetime multiplication and division.

2.4 Let $z = x + jt$. Find (a) z^2, (b) z^3, (c) z^4, (d) z^5.

2.5 Show that

$$(x+jt)^n = \sum_{k=0}^{[n/2]} \binom{n}{2k} x^{n-2k} t^{2k} + j \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} x^{n-2k-1} t^{2k+1} .$$

3. Spacetime Plane and the three types of spacetime numbers.

We recall that complex numbers are visualized as points or as vectors in the $z = x + iy$ plane. In the same way we can visualize spacetime numbers in a $z = x + jt$ plane. We construct the space-time plane by labeling the horizontal axis as the space axis and the vertical axis as our time axis.

When we divide by $z = x + jy$ (see Table 1 above), we obtain the expression $x^2 - t^2$ in the denominator. This quantity puts a restriction on the division, for if $x^2 - t^2 = 0$, then the division is not possible. We expect division by zero to be impossible, but here division by certain non-zero numbers is also impossible. This is a difficulty which does not occur in complex division because (see Table 1) the denominator is $x^2 + y^2$. It will be useful to subdivide the numbers in the spacetime plane into the following three distinct types:

Case I: $x^2 - t^2 = 0, \quad x^2 = t^2, \quad \text{Light-like}$

Case II: $x^2 - t^2 > 0, \quad x^2 > t^2, \quad \text{Space-like}$

Case III: $x^2 - t^2 < 0, \quad x^2 < t^2, \quad \text{Time-like}$

If $x^2 = t^2$, we can graph these two lines $x = t$ and $x = -t$ in the space-time plane (see Figure 1). We call these *light-like lines*. Any spacetime numbers lying on these lines will be referred to as *light-like numbers.*

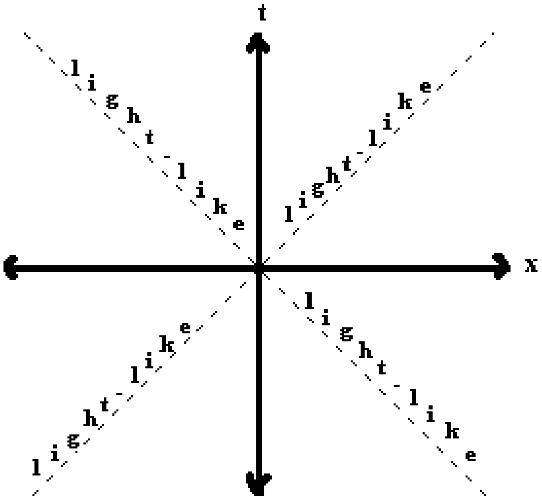

Figure 1: Light-like numbers

Now we examine Case II when $x^2 > t^2$. This corresponds to the shaded areas in Figure 2. We refer to these areas as the space-like regions and thus, any numbers in these regions are *space-like numbers*.

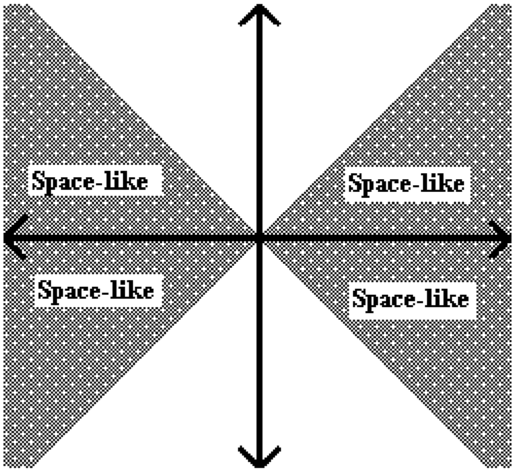

Figure 2: Space-like numbers

Examining Case III, when $x^2 < t^2$, we see that this corresponds to the shaded areas in Figure 3. We refer to these areas as the time-like regions and thus, any number in these regions are *time-like numbers.*

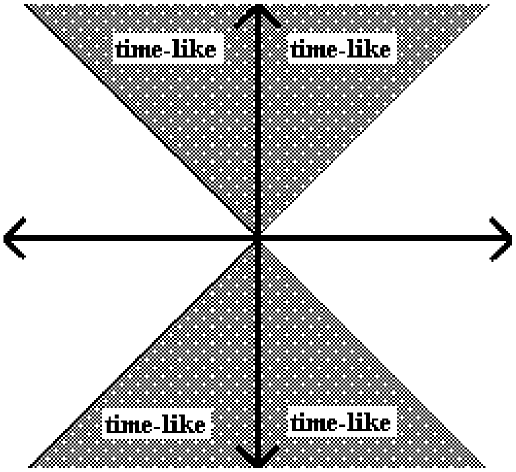

Figure 3: Time-like numbers

In the next section we will show one reason why we have distinguished between these three regions.

Problem

(3.1) Identify each of the following spacetime numbers as light-like, space-like or time-like. (a) $3 - 2j$, (b) $-4 + 7j$, (c) $2 - 2j$, (d) $\pi + \pi^2 j$, (e) 0, (f) 7 , (g) $-ej$.

4. An unexpected feature of multiplication.

Unlike complex numbers, our spacetime numbers have special features under multiplication. For example, when two space-like numbers are multiplied, the product is a space-like number. The full story is summarized in the following table:

Multiplication	**Light-like**	**Space-like**	**Time-like**
Light-like	Light-like	Light-like	Light-like
Space-like	Light-like	Space-like	Time-like
Time-like	Light-like	Time-like	Space-like

Table 2: Results of spacetime multiplication

We can easily verify any of these entries. Let's take an example of multiplying a light-like number and a time-like number. Remember that a light-like number is one in which $x^2 = t^2$, and a time-like number is one in which $x^2 < t^2$. We arbitrarily choose our light-like number to be $z_1 = -3 + 3j$ and our time-like number to be $z_2 = 4 + 6j$. Now we must perform the multiplication.

$$z_1 \times z_2 = (-3 + 3j) \times (4 + 6j)$$

$$= (-12 + 18) + (-18j + 12j)$$

$$= 6 - 6j$$

This is indeed a light-like number because $x = 6$ and $t = -6$, so $x^2 = t^2$.

The multiplication table we have generated resembles multiplication of real numbers. If we simply think of real numbers as zero, positive, or negative, we can construct a table similar to the one for our space-time number multiplication.

Multiplication	**Zero**	**Positive**	**Negative**
Zero	Zero	Zero	Zero
Positive	Zero	Positive	Negative
Negative	Zero	Negative	Positive

Table 3: Real number multiplication

Comparing Tables 2 and 3 we see that light-like numbers behave in a certain sense like the real number zero. Zero times anything is zero, and light-like times anything is light-like. In a similar way we see that negative and positive real numbers interact under multiplication is the same way as time-like and space-like numbers. This analogy helps our memory. Now it becomes easy to recall that multiplication of two time-like numbers produces a space-like number, just as two negative numbers produce a positive number when multiplied together.

We will now show how to verify in general that the product of any space-like number with a time-like number is time-like. We begin by denoting the numbers by

(4.1) $z = A + aj$ which is a space-like number $A^2 > a^2$

(4.2) $Z = b + Bj$ which is a time-like number $B^2 > b^2$

If we multiply the two spacetime numbers together, we obtain:

$$\begin{aligned} zZ &= (A+aj)(b+Bj) \\ &= Ab + ABj + abj + aBj^2 \\ &= (Ab+aB) + (AB+ab)j \end{aligned}$$

Let us compare the square of the space part $(Ab + aB)$ of our product , and the square of the time part $(AB + ab)$. When we subtract these squares we get:

$(space\,part)^2 - (time\,part)^2 =$

$$\begin{aligned} (Ab+aB)^2 - (AB+ab)^2 &= (A^2b^2 + 2AaBb + a^2B^2) - (A^2B^2 + 2AaBb + a^2b^2) \\ &= A^2b^2 + a^2B^2 - A^2B^2 - a^2b^2 \\ &= A^2b^2 - A^2B^2 + a^2B^2 - a^2b^2 \\ &= (A^2b^2 - A^2B^2) - (a^2b^2 - a^2B^2) \\ &= A^2(b^2 - B^2) - a^2(b^2 - B^2) \\ &= (A^2 - a^2)(b^2 - B^2) \end{aligned}$$

But . $A^2 > a^2$ and $B^2 > b^2$, which makes the last product above negative Thus we have demonstrated that a space-like number multiplied by a time-like number will yield a time-like number.

Readers having some familiarity with abstract algebra will recognize that the spacetime numbers do not form a field as does our complex number system. The spacetime numbers are not an integral domain. This is because there are spacetime numbers other than zero that do not have inverses. They are the light-like numbers. The spacetime number system is an example of a commutative ring. The light-like numbers are called divisors of zero.

Problems

(4.1) Show that the product of two space-like numbers is space-like.

(4.2) Show that the product of two time-like numbers is space-like.

(4.3) Show that the product of any number with a light-like number is light-like.

5. Zeroes of polynomials.

Suppose we are given a polynomial such as

$$x^2 - 4x + 3 = 0 . \tag{5.1}$$

We can solve for the roots to obtain $x = 2 \pm 1$. This corresponds to $x = 3$ and $x = 1$ Upon closer examination of $x = 2 \pm 1$, (since $j = \pm 1$) we might expect that $x = 2 + j$ and $x = 2 - j$ are also roots. Let's check $x = 2 + j$.

$$x^2 - 4x + 3 = 0$$

$$(2 + j)^2 - 4(2 + j) + 3 = 0$$

$$4 + 2j + 2j + j^2 - 8 - 4j + 3 = 0$$

$$0 = 0$$

Thus, $x = 2 + j$ is a zero of the polynomial $x^2 - 4x + 3 = 0$. The reader can verify that $x = 2 - j$ is also a zero of the above polynomial.

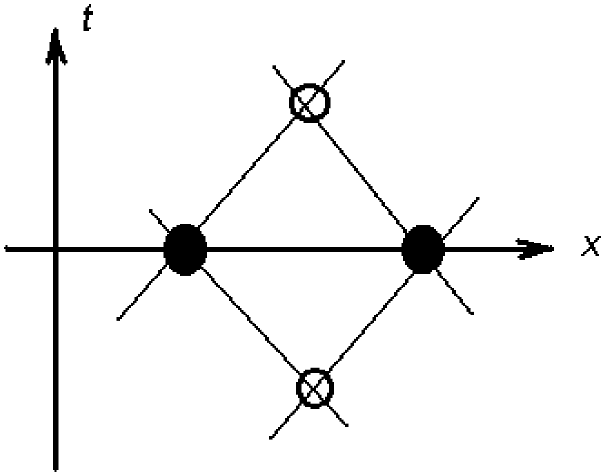

Figure 4: Two spacetime roots corresponding to a pair of real roots

What this means is that in the spacetime plane, there are a total of four roots to a quadratic equation which has two real roots. For simplicity, if we take any polynomial which possesses only real roots, it becomes easy to calculate the remaining spacetime roots. Every pair of real roots contributes a pair of spacetime roots (see Fig. 4) in the manner described above for (5.1). We begin by plotting all the real roots on the x-axis of the spacetime plane. Then we draw lines with slopes equal to $+1$ and -1 through the plotted root points. Where these lines intersect corresponds to the remaining spacetime roots of the polynomial. In Figure 5 we see the case of a quartic polynomial with four real roots. Our construction results in a new pair of spacetime roots for each possible pair of real roots. A total of 16 roots for this quartic are found in the spacetime plane.

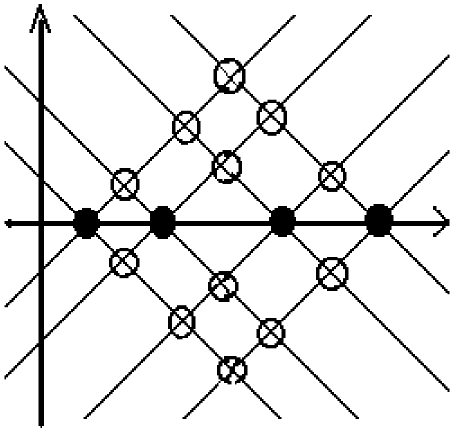

Figure 5: The Spacetime Roots of a Fourth Degree Polynomial

It can be shown that for any polynomial which possesses n real roots, there exists n^2 spacetime roots. We could continue by examining what happens when complex roots occur and other matters, but choose to stop here.

Problems

Find all spacetime roots of

(5.1) $x^2 - 1 = 0$

(5.2) $x^2 - 4x = 0$

(5.3) $x^2 - 2bx + c = 0$

(5.4) $x^3 - 4x = 0$

(5.5) $x^4 - 5x^2 + 4 = 0$

6. Comments on the references and possible future study.

We plan to submit two more papers in the same easy style as this one. The first will discuss functions of a spacetime variable, in particular the exponential. The second paper will show applications to special relativity.

The best reference known to us for further study is the excellent paper by Garret Sobczyk [23]. We strongly recommend this paper even though it does use the terminology of modern abstract algebra. We also recommend the paper by Fjelstad [4]. In this paper Fjelstad explains how he and his students rediscovered spacetime numbers (he calls them perplex numbers). The references by Band [1], Majernik [14] and Ronveaux [18] are letters in response to the paper of Fjelstad and we think the reader will find these very interesting. Readers familiar with fractals and the Mandelbrot set will find the papers of Senn [19] and Metzier [15] interesting because they show how the Mandelbrot set looks in the spacetime plane. The paper by Lambert [8] is an amusing critique of this entire enterprise (in French). References [3, 5, 7, 9] are graduate level text books. The remaining references are research papers in the theory and application of spacetime numbers and other related number systems.

The list of texts and papers below represents only a small part of the available literature. Notice that any material on Clifford algebras is of potential interest since spacetime numbers are a special case of these.

There is as yet, no standard terminology for the items studied in this subject. We end this paper with a list of names and symbols that have been used in the literature:

(1) *Spacetime numbers.*

These are called *hyperbolic numbers* by Sobczyk [23], *hallucinatory numbers* by Fjelstad [4] and Metzier [15], *binary numbers* by Majernik [11], [12], [13], and *hyperbolic complex numbers* by Zhong [25] and by Wu and Zhong [24].

(2) *The spacetime unit j*

This is called the *unipotent u* by Sobczyk [23], the *"hallucinatory " number h* by

Fjelstad [4] and Metzier [15], an *"imaginary" unit* ε *or* λ by Majernik [11], [12], [13] and the *hyperbolic imaginary unit* ε by Zhong [25] and by Wu and Zhong [24].

7. Answers to selected problems

2.1 (a) $-2+10j$, (b) $8+4j$, (c) $6-26j$, (d) $-\dfrac{9+11j}{4}$

2.2 The division is impossible because of division by zero.

2.4 (a) x^2+t^2+2xtj, (b) $(x^3+3xt^2)+(3x^2t+t^3)j$, (c) $(x^4+6x^2t^2+t^4)+(4x^3t+4xt^3)j$

(d) $(x^5+10x^3t^2+5xt^4)+(5x^4t+10x^2t^3+t^5)j$.

3.1 (a) space-like, (b) time-like, (c) light-like, (d) time-like, (e) light-like, (f) space-like, (g) time-like.

(5.1) $\pm 1, \pm j$.

(5.2) $0, 4, 2\pm 2j$,

(5.3) $b\pm\sqrt{b^2-c}, b\pm\sqrt{b^2-c}\,j$

(5.4) $0, \pm 2, 1\pm j, -1\pm j, \pm 2j..$

(5.5) $\pm 1, \pm 2, \pm j, \pm 2j, (3\pm j)/2, (-3\pm j)/2, (1\pm 3j)/2, (-1\pm 3j)/2$

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