TAYLOR'S SERIES GENERALIZED FOR FRACTIONAL DERIVATIVES AND APPLICATIONS*

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Abstract. The familiar Taylor's series expansion of the function $f(z)$ has for its general term $D^n f(z_0)(z - z_0)^n/n!$. A new generalization of Taylor's series in which the general term is $D^{an+\gamma} f(z_0)(z - z_0)^{an+\gamma}/\Gamma(an + \gamma + 1)$, where $a > 0$ and γ is an arbitrary complex number, is examined. This new series is extended further to a form which includes the familiar Lagrange's expansion as a special case. The derivatives appearing in this series are of order $an + \gamma$ and are called "fractional derivatives." Examples of the use of this new series for discovering generating functions are given.

1. Introduction. A fractional derivative $D^{\alpha}_{g(z)} f(z)$ is an extension of the familiar nth derivative $D^n_{g(z)} f(z) = d^n f(z)/(dg(z))^n$ of the function $f(z)$ with respect to $g(z)$ to nonintegral values of n. The literature contains many examples of the use of fractional derivatives in the solution of problems in ordinary differential equations [8], partial differential equations [4], [13] and integral equations [3].

The study of the special functions of mathematical physics is also facilitated by the introduction of fractional differential operators. Consider, for example, the various representations of the Bessel function $J_v(z)$ of order v. Power series and definite integral representations are the most common; however, the less familiar derivative representation

$$J_v(z) = \pi^{-1/2}(2z)^{-v} D_{z^2}^{-v-1/2} \frac{\cos z}{z} \tag{1.1}$$

warrants further attention. When $-v - 1/2$ is a natural number, (1.1) reveals that $D_{z^2}^{-v-1/2}$ is the usual elementary differential operator, and thus $J_v(z)$ is an elementary function. When $-v - 1/2$ is not a natural number, the operator $D_{z^2}^{-v-1/2}$ still behaves very much like the familiar differential operator from the elementary calculus. The operation $D^a D^b = D^{a+b}$, the Leibniz rule [9], [10], the chain rule [9], [11] and other generalizations of the manipulations so familiar from the elementary calculus are valid for nonintegral values of a and b. These manipulations permit us to find easily many relations for the special functions from representations similar to (1.1) which would not otherwise seem obvious [7], [8], [9], [10], [11]. Table 1 gives a short list of fractional derivative representations for the special functions.

In this paper the Taylor's series is generalized to include fractional derivatives and thus provides an additional tool which is particularly convenient for the study of the special functions through their fractional derivative representations. There are two equivalent forms of our general result:

$$\begin{aligned} &\sum_{k \in K} a^{-1} \omega^{-\gamma k} f(\theta^{-1}(\theta(z)\omega^k)) \\ &= \sum_{n=-\infty}^{\infty} \frac{D^{an+\gamma}_{z-b}[f(z)\theta'(z)[(z - z_0)/\theta(z)]^{an+\gamma+1}]|_{z=z_0}\, \theta(z)^{an+\gamma}}{\Gamma(an + \gamma + 1)} \end{aligned} \tag{1.2}$$

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and

$$\text{(1.2a)} \qquad \sum_{k\in K} a^{-1}\omega^{-\gamma k} f(\theta^{-1}(\theta(z)\omega^k)) = \sum_{n=-\infty}^{\infty} \frac{\theta(z)^{an+\gamma}}{2\pi i} \int_b^{(z_0^+)} \frac{f(\xi)\theta'(\xi)\,d\xi}{\theta(\xi)^{an+\gamma+1}}.$$

There are several restrictions which must be imposed on the functions and parameters in (1.2) and (1.2a), all of which are listed in the hypothesis of Theorem 4.1. For the moment, it suffices to notice the following:

(i) The order of the derivatives in (1.2) is $an + \gamma$, where n is the integral index of summation, $a > 0$, and γ is an arbitrary complex number.

(ii) b is a fixed point in the z-plane and $\{z \mid |\theta(z)| = |\theta(b)|\}$ defines a simple closed curve C on which the series (1.2) and (1.2a) converge. $\theta(z)$ is an analytic function inside and on C. $\theta(z)$ has only one zero inside C, located at $z = z_0$, and that zero is simple.

(iii) $\omega = \exp(2\pi i/a)$, and the finite set of integers K is defined by $K = \{k \mid k$ is integral, and $\arg\theta(b) < \arg\theta(z) + 2\pi k/a < \arg\theta(b) + 2\pi\}$.

While the general formulas (1.2) and (1.2a) are new, several special cases are familiar from the literature.

Case 1. If $a = 1, \gamma = 0$ and $\theta(z) = z - z_0$ in (1.2), we have the familiar Taylor's series

$$f(z) = \sum_{n=0}^{\infty} D^n f(z_0)(z - z_0)^n/n!.$$

Case 2. We obtain Lagrange's expansion [16, p. 132] from (1.2) (after an integration by parts) by taking $a = 1, \gamma = 0$ and $\theta(z) = \theta_1(z)(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} D^{n-1}\{f'(z_0)/\theta_1(z_0)^n\}\,\theta(z)^n/n!$$

Case 3. If we take $a = 1$ and $\gamma = 0$ in (1.2a), we obtain Teixeira's extended form of Burmann's theorem [16, p. 131]:

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{\theta(z)^n}{2\pi i} \oint \frac{f(\xi)\theta'(\xi)\,d\xi}{\theta(\xi)^{n+1}}.$$

Case 4. We obtain the least familiar special case of (1.2) which can be found in the literature by taking $a = 1, \theta(z) = z - z_0$, and γ arbitrary. It is called the Taylor–Riemann series:

$$\text{(1.3)} \qquad f(z) = \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{n+\gamma} f(z)|_{z=z_0}(z - z_0)^{n+\gamma}}{\Gamma(n + \gamma + 1)}.$$

This series was first considered formally by Riemann [12] in 1847, in a manuscript probably never intended for publication. Riemann did not prove (1.3), but used its structure to suggest a definition of fractional differentiation. The special cases of (1.3) in which $f(z)$ is e^z and z^p were studied by Heaviside [6, Chap. 7, 8] and Watanabe [14]. The first critical discussion of (1.3) for arbitrary functions $f(z)$ was not given until 1945 when G. H. Hardy [5] considered (1.3) as an asymptotic expansion of $f(z)$ and as a series summable Borel to $f(z)$. The first analysis of the pointwise convergence of the series (1.3) to the function $f(z)$ in the z-plane seems to be [9, Chap. 3]. The nature of the pointwise convergence of (1.3) in the z-plane is given as a special case of the more general formula (1.2) in Theorem 4.1 of this paper.

If we restrict a to the interval $0 < a \leqq 1$, the left-hand side of (1.2) contains only the term in which $k = 0$, and we obtain the particularly simple series

$$f(z)a^{-1} = \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{an+\gamma}[f(z)\theta'(z)[(z - z_0)/\theta(z)]^{an+\gamma+1}]|_{z=z_0}\theta(z)^{an+\gamma}}{\Gamma(an + \gamma + 1)}.$$

To the best of the author's knowledge, neither this series nor the more general series (1.2) have appeared before in the literature.

Finally, a few examples of the generalized Taylor's series are studied for specific functions $f(z)$. We find that (1.2) is particularly useful for obtaining generating functions for the special functions of mathematical physics when these special functions are represented by fractional derivatives.

2. Fractional derivatives and special functions. In this section we review the definition of fractional differentiation and give examples of common special functions of mathematical physics represented by fractional derivatives of elementary functions.

The most common definition for the fractional derivative of $f(z)$ of order α found in the literature is the "Riemann–Liouville integral" [2], [3], [4], [5], [7], [8], [9], [10], [11], [13], [14]:

$$D_z^\alpha f(z) = \Gamma(-\alpha)^{-1} \int_0^z f(t)(z - t)^{-\alpha-1}\, dt$$

where $\mathrm{Re}(\alpha) < 0$. The concept of a fractional derivative $D_{g(z)}^\alpha f(z)$ with respect to an arbitrary function $g(z)$ was apparently introduced for the first time in the author's papers [9], [10], while the idea appeared earlier for certain specific functions $g(z)$ in [4]. The most convenient form of the definition for our purposes is given through a generalization of Cauchy's integral formula. A thorough motivation for the following precise definition is found in [9], [10].

DEFINITION 2.1. Let $f(z)$ be analytic in the simply connected region R. Let $g(z)$ be regular and univalent on R, and let $g^{-1}(0)$ be an interior or boundary point of R. Assume also that $\oint_C f(z)\, dz = 0$ for any simple closed contour C in $RU\{g^{-1}(0)\}$ through $g^{-1}(0)$. Then if α is not a negative integer and z is in R, we define the *fractional derivative of order* α of $f(z)$ with respect to $g(z)$ to be

$$D_{g(z)}^\alpha f(z) = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{g^{-1}(0)}^{(z^+)} \frac{f(\zeta)g'(\zeta)\, d\zeta}{(g(\zeta) - g(z))^{\alpha+1}}. \tag{2.1}$$

For nonintegral α, the integrand has a branch line which begins at $\zeta = z$ and passes through $\zeta = g^{-1}(0)$. The notation on this integral implies that the contour of integration starts at $g^{-1}(0)$, encloses z once in the positive sense, and returns to $g^{-1}(0)$ without cutting the branch line or leaving $RU\{g^{-1}(0)\}$. (See Fig. 1.)

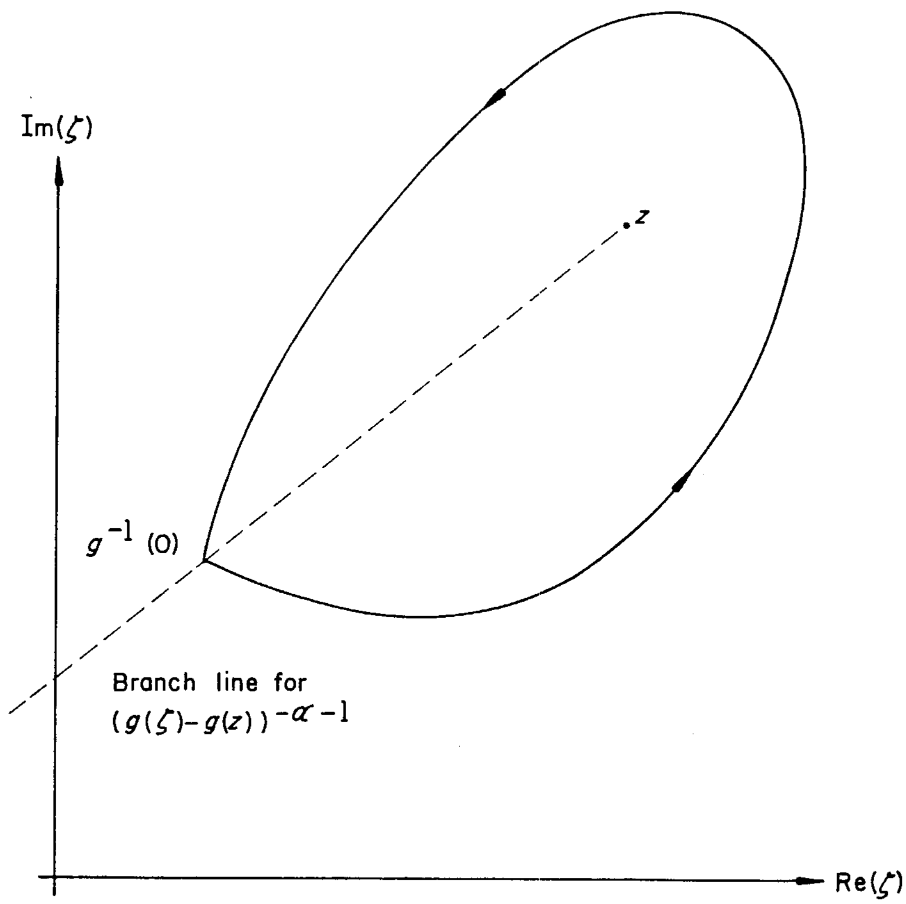

FIG. 1. *Branch line and contour of integration for the Definition* 2.1 *of fractional differentiation*

It is particularly interesting to set $g(z) = z - a$, for we find that

(2.2) $$D^{\alpha}_{z-a} f(z) = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{a}^{(z+)} f(\zeta)(\zeta - z)^{-\alpha - 1}\, d\zeta.$$

While ordinary derivatives with respect to z and $z - a$ are equal, (2.2) shows that this is not the case for fractional derivatives, since the value of the contour integral depends on the point $\zeta = a$ at which the contour crosses the branch line.

The equivalence of the two forms of the generalized Taylor's series (1.2) and (1.2a) is seen at once from (2.2).

Contour integrals of the type (2.1) occur often in the representations of special functions. Table 1 gives a brief list of fractional derivative representations for a few selected functions. These are particularly convenient for use with the generalized Taylor's series (1.2). Fractional derivative representations of special functions are also found in [8] and can be easily constructed from the tables in [2].

3. Motivation for the generalized Taylor's theorem. The generalized Taylor's theorem features a "finite sum over k" on the left-hand side of (1.2). Why? An intuitive answer to this question is provided in this section through a formal examination of (1.2) in the special case in which a and γ are integers and $\theta(z) = z$. The relationship between the generalized Taylor's series and the Fourier series is then suggested by the consideration of a second formal example in which

TABLE 1

Special functions expressed as fractional derivatives

Name	Derivative Representation
Hypergeometric function	$F(a,b;c;z) = \dfrac{\Gamma(c)z^{1-c}}{\Gamma(b)} D_z^{b-c} z^{b-1}(1-z)^{-a}$
Confluent hypergeometric function	${}_1F_1(a;c;z) = \dfrac{\Gamma(c)z^{1-c}}{\Gamma(a)} D_z^{a-c} e^z z^{a-1}$
Bessel function	$J_v(z) = \pi^{-1/2}(2z)^{-v} D_{z^2}^{-v-1/2} \dfrac{\cos z}{z}$
Modified Bessel function	$I_v(z) = \pi^{-1/2}(2z)^{-v} D_{z^2}^{-v-1/2} \dfrac{\cosh z}{z}$
Struve function	$\mathbf{H}_v(z) = \pi^{-1/2}(2z)^{-v} D_{z^2}^{-v-1/2} \dfrac{\sin z}{z}$
Modified Struve function	$\mathbf{L}_v(z) = \pi^{-1/2}(2z)^{-v} D_{z^2}^{-v-1/2} \dfrac{\sinh z}{z}$
Legendre function of the first kind	$P_v(z) = D_{1-z}^{v}(1-z^2)^v/(\Gamma(v+1)2^v)$
Associated Legendre function of the first kind	$P_v^u(z) = (1-z^2)^{u/2} D_{1-z}^{v}(1-z^2)^v/(\Gamma(v+1)2^v)$
Laguerre function	$L_v^{(a)}(z) = \dfrac{\Gamma(a+v+1)z^{-a}}{\Gamma(v+1)\Gamma(-v)} D_z^{-a-v-1} e^z z^{-v-1}$
Incomplete gamma function	$\gamma(a,z) = \Gamma(a)e^{-z} D_z^{-a} e^z$

$0 < a \leqq 1$. Together, these two examples provide intuitive insight into the structure of the generalized Taylor's series and give preliminary assurance of its validity. The complete proof is postponed until the next section.

Case 1. Let $\theta(z) = z$ and a and γ be integers in (1.2). We then obtain

$$\sum_{k=0}^{a-1} \omega^{-\gamma k} f(z\omega^k) = a \sum_{n=0}^{\infty} f_{an+\gamma} z^{an+\gamma}, \tag{3.1}$$

where we have written $f_{an+\gamma}$ for $D^{an+\gamma}f(0)/(an+\gamma)!$, and $\omega = \exp(2\pi i/a)$. The examination of the special case in which $a = 3$ and $\gamma = 1$ is sufficient to suggest the manner in which the general case proceeds:

$$\begin{aligned} f(z) &= f_0 + f_1 z + f_2 z^2 + f_3 z^3 + f_4 z^4 + \cdots. \\ \omega^{-1} f(z\omega) &= \omega^{-1} f_0 + f_1 z + \omega f_2 z^2 + \omega^2 f_3 z^3 + f_4 z^4 + \cdots, \\ \omega^{-2} f(z\omega^2) &= \omega^{-2} f_0 + f_1 z + \omega^2 f_2 z^2 + \omega^4 f_3 z^3 + f_4 z^4 + \cdots. \end{aligned}$$

Summing these columns we see at once that the right-hand side is $3\sum_{n=0}^{\infty} f_{3n+1} z^{3n+1}$

as (3.1) predicts. Equation (3.1) is true for arbitrary integral a and γ by an equivalent calculation. This example shows that the finite sum over k in the generalized Taylor's theorem is natural and to be expected. (If we think of the way in which $\cosh(z)$ is related to e^z, we see at once that this is the special case of (3.1) in which $a = 2$, $\gamma = 0$, and $f(z) = e^z$.)

Case 2. Let $0 < a \leqq 1$ in (1.2a). We then have

$$\theta(z)^{-\gamma} f(z) = \sum_{n=-\infty}^{\infty} \frac{a}{2\pi i} \int_b^{(z_0^+)} \frac{f(\xi)\theta'(\xi)\, d\xi}{\theta(\xi)^{an+\gamma+1}} \theta(z)^{an}. \tag{3.2}$$

Let $\theta(\xi) = |\theta(b)| \exp(i\phi_0)$, $\theta(z) = |\theta(b)| \exp(i\phi)$, and $\theta(z)^{-\gamma} f(z) = F(\phi)$ in (3.2) and we obtain

$$F(\phi) = \sum_{n=-\infty}^{\infty} \frac{a}{2\pi} \int_{\arg\theta(b)}^{\arg\theta(b)+2\pi} F(\phi_0) \exp(-ian\,\phi_0)\, d\phi_0 \exp(ian\,\phi). \tag{3.3}$$

The right-hand side of (3.3) is the Fourier series expansion of the function

$$F_0(\phi) = \begin{cases} F(\phi) & \text{for } \arg\theta(b) < \phi < \arg\theta(b) + 2\pi, \\ 0 & \text{otherwise} \end{cases}$$

over the interval $|\phi - \arg\theta(b) - \pi| < \pi/a$. Since we are only interested in ϕ such that $\arg\theta(b) < \phi < \arg\theta(b) + 2\pi$, formula (3.3) is valid. This example reveals that the generalized Taylor's theorem for arbitrary nonintegral a is nothing more than the Fourier series of a new function constructed from the function $f(z)$.

A rigorous proof of the generalized Taylor's theorem for arbitrary positive a could be constructed from the Fourier series analysis just given. However, a simpler method employing contour integration (not unlike the usual proof of Laurent's theorem) is given in the next section.

4. Proof of the generalized Taylor's series. Having examined examples which give motivation for the structure of the generalized Taylor's series (1.2) and (1.2a), we proceed to a rigorous derivation.

THEOREM 4.1. *Let a be real and positive, and let $\omega = \exp(2\pi i/a)$. Let $\theta(z)$ be a given function such that* (i) *the curves $C(r) = \{z \mid |\theta(z)| = r\}$ are simple and closed for each r such that $0 < r \leqq p$,* (ii) *$\theta(z)$ is analytic inside and on $C(p)$, and* (iii) *$\theta(z)$ has only one zero inside $C(p)$ and that zero is a simple one located at $z = z_0$. Let $b \neq z_0$ be a fixed point inside $C(p)$. Let $\theta(z)^q = \exp(q \ln\theta(z))$ denote that branch of the function which is continuous and single-valued on the region inside $C(p)$ cut by the branch line $z = z_0 + (b - z_0)r$, $0 \leqq r$, such that $\ln\theta(z)$ is real where $\theta(z) > 0$. Let $f(z)$ satisfy the conditions of Definition* 2.1 *for the existence of $D^{\alpha}_{z-b} f(z)$ for $\{z \mid z$ inside $C(p)$; but $z \neq b + r\exp(i\arg(b - z_0))$, $0 \leqq r\}$. Let $K = \{k \mid k$ integral, and $\arg\theta(b) < \arg\theta(z) + 2\pi k/a < \arg\theta(b) + 2\pi\}$. Then for arbitrary γ and z on $\{z \mid z$ on the curve $C(|\theta(b)|)$, but $\theta(z)^a \neq \theta(b)^a\}$, the generalized Taylor's series* (1.2) *and* (1.2a) *are valid.*

Proof. The maximum modulus theorem insures that the set of simple closed curves $C(r)$, $0 < r \leqq p$, are such that $C(s)$ is contained inside $C(t)$ for $s < t$. Let C_x denote the contour consisting of a straight line segment from $\xi = b$ to $\xi = b + x(b - z_0)$, the curve $C(|\theta(b + x(b - z_0))|)$ traversed once in the positive

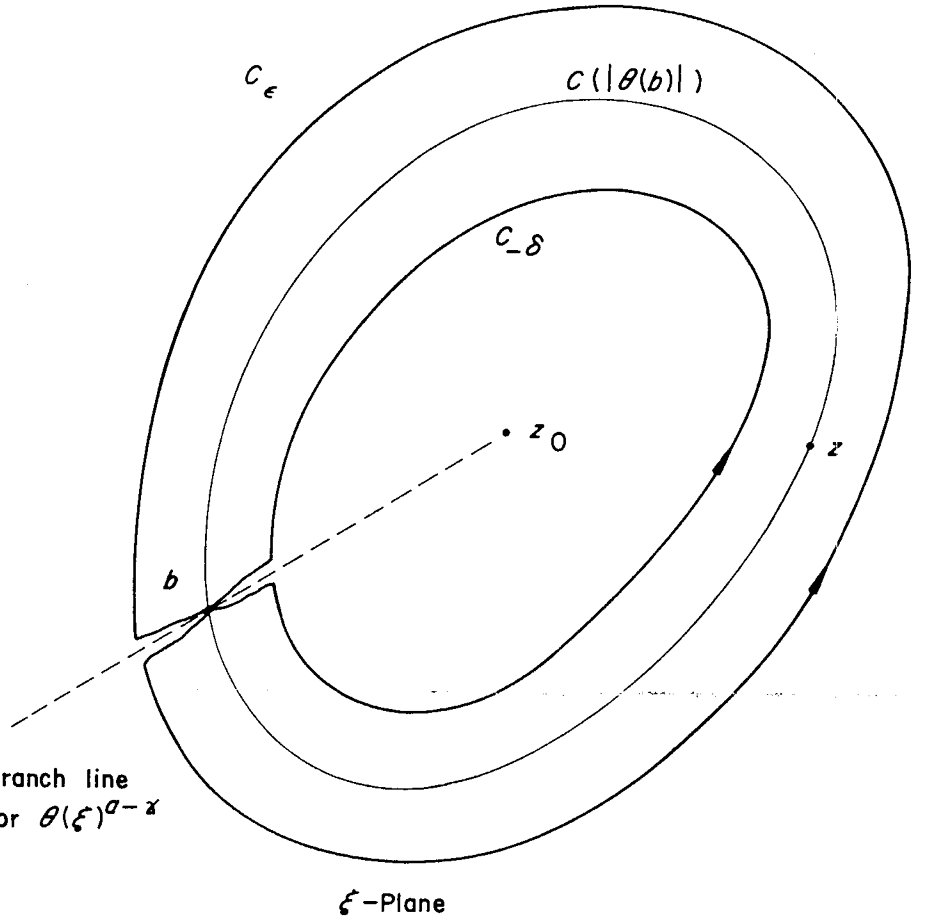

FIG. 2. *Contours of integration used in the proof of Theorem* 4.1

sense, and a straight line segment from $b + x(b - z_0)$ back to b. The contours C_ε and $C_{-\delta}$ are shown in Figure 2.

Consider the integral

$$I = \frac{\theta(z)^\gamma}{2\pi i} \int_{C_\varepsilon - C_{-\delta}} \frac{\theta(\xi)^{a-\gamma-1}\theta'(\xi) f(\xi)\, d\xi}{\theta(\xi)^a - \theta(z)^a}. \tag{4.1}$$

The integrand in (4.1) contains poles at the points where $\theta(\xi)^a = \theta(z)^a$. This means that $\theta(\xi) = \theta(z) \exp(2\pi k i/a)$ for $k \in K$. (The set of integers K is defined in the hypothesis.) Thus there are poles at $\xi = \theta^{-1}(\theta(z)\omega^k)$, $k \in K$, while the integrand of (4.1) is analytic for all other values of ξ inside the closed contour $C_\varepsilon - C_{-\delta}$. Each of these poles is simple because the number of roots of the equation $\theta(\xi) = c$, $|c| < p$, is given by the argument principle as

$$(2\pi i)^{-1} \int_{C(p)} \frac{\theta'(\xi)\, d\xi}{\theta(\xi) - c} = \sum_{n=0}^{\infty} (2\pi i)^{-1} c^n \int_{C(p)} \theta'(\xi)\theta(\xi)^{-n-1}\, d\xi.$$

All terms in this last sum vanish but the first, which equals 1 since $\xi = z_0$ is a simple root of θ. The residue at $\xi = \theta^{-1}(\theta(z)\omega^k)$ is given by

$$\lim_{\xi \to \theta^{-1}(\theta(z)\omega^k)} \left\{ \frac{(\xi - \theta^{-1}(\theta(z)\omega^k))\theta(\xi)^{a-\gamma-1}\theta'(\xi) f(\xi)}{\theta(\xi)^a - \theta(z)^a} \right\};$$

and using l'Hospital's rule this becomes

$$\left.\frac{\theta(\xi)^{a-\gamma-1}f(\xi)}{a\theta(\xi)^{a-1}}\right|_{\xi=\theta^{-1}(\theta(z)\omega^k)} = \frac{f(\theta^{-1}(\theta(z)\omega^k))}{a\omega^{\gamma k}\theta(z)^\gamma}.$$

Thus we see from the residue theorem that

$$I = \sum_{k\in K} a^{-1}\omega^{-\gamma k} f(\theta^{-1}(\theta(z)\omega^k)). \tag{4.2}$$

Returning to (4.1) we see that

$$\begin{aligned} I &= \frac{\theta(z)^\gamma}{2\pi i}\left\{\int_{C_\varepsilon} - \int_{C_{-\delta}}\right\} \\ &= \frac{\theta(z)^\gamma}{2\pi i}\left\{\int_{C_\varepsilon} \frac{\theta(\xi)^{-\gamma-1}\theta'(\xi)f(\xi)\,d\xi}{1-[\theta(z)/\theta(\xi)]^a} + \int_{C_{-\delta}} \frac{\theta(\xi)^{a-\gamma-1}\theta'(\xi)f(\xi)\,d\xi}{\theta(z)^a[1-[\theta(\xi)/\theta(z)]^a]}\right\}. \end{aligned}$$

Expanding the denominators of both integrals in powers of $[\theta(z)/\theta(\xi)]^a$ we obtain

$$\begin{aligned} I = \frac{\theta(z)^\gamma}{2\pi i}\Bigg\{ &\sum_{n=0}^{N} \int_{C_\varepsilon} \theta(\xi)^{-an-\gamma-1}\theta'(\xi)f(\xi)\,d\xi\,\theta(z)^{an} \\ &+ \sum_{n=-1}^{-N} \int_{C_{-\delta}} \theta(\xi)^{-an-\gamma-1}\theta'(\xi)f(\xi)\,d\xi\,\theta(z)^{an} \\ &+ R_\varepsilon(N) + R_{-\delta}(N)\Bigg\}, \end{aligned} \tag{4.3}$$

where

$$R_\varepsilon(N) = \int_{C_\varepsilon} \frac{\theta(\xi)^{-\gamma-1}[\theta(z)/\theta(\xi)]^{aN+a}\theta'(\xi)f(\xi)\,d\xi}{1-[\theta(z)/\theta(\xi)]^a}$$

and

$$R_{-\delta}(N) = \int_{C_{-\delta}} \frac{\theta(\xi)^{a-\gamma-1}[\theta(\xi)/\theta(z)]^{aN}\theta'(\xi)f(\xi)\,d\xi}{1-[\theta(\xi)/\theta(z)]^a}.$$

We note that the regularity of θ and f permits us to deform the contours of integration C_ε and $C_{-\delta}$ in (4.3) provided the contours start and end at $\xi = b$ and do not cross the branch line for $\theta(\xi)^{a-\gamma}$ (see Fig. 2). Comparison of (4.3) with the definition of fractional derivative

$$\left. D_{z-b}^{an+\gamma}F(z)\right|_{z=z_0} = \frac{\Gamma(an+\gamma+1)}{2\pi i}\int_b^{(z_0^+)} \frac{F(\xi)\,d\xi}{(\xi-z_0)^{an+\gamma+1}}$$

yields at once

$$\begin{aligned} I = &\sum_{n=-N}^{N} \frac{D_{z-b}^{an+\gamma}[f(z)\theta'(z)((z-z_0)/\theta(z))^{an+\gamma+1}]|_{z=z_0}\,\theta(z)^{an+\gamma}}{\Gamma(an+\gamma+1)} \\ &+ \theta(z)^\gamma \frac{R_\varepsilon(N)+R_{-\delta}(N)}{2\pi i}. \end{aligned} \tag{4.4}$$

$R_\varepsilon(N)$ is the sum of three integrals, two over short line segments of length ε and one over the contour $C(|\theta(b + \varepsilon(b - z_0))|)$. Since the integrand contains the term $[\theta(z)/\theta(\xi)]^a$ (which has modulus less than 1 if z is on $C(|\theta(b)|)$) to the power $N + 1$, it is easy to see that for sufficiently small ε and large N, $R_\varepsilon(N)$ can be made arbitrarily small. A similar argument holds for $R_{-\delta}(N)$. Comparing (4.2), (4.3) and (4.4) we see that the theorem is proved.

If a is a natural number, the generalized Taylor's series sometimes converges in a region larger than that described in Theorem 4.1. This special case is examined in the following corollary.

COROLLARY 4.1. *Assume a is a natural number in the hypothesis of the previous theorem and that $f(z) = (z - b)^{\gamma+N}g(z)$, where $g(z)$ is analytic for z inside $C(p)$ and N is an integer. Then the generalized Taylor's series* (1.2) *converges not only for z on $C(|\theta(b)|)$, $\theta(z)^a \neq \theta(b)^a$, but also for all z in the ring-shaped region between $C(p)$ and $C(|\theta(b)|)$.*

Proof. The integrand of I in (4.1) is

$$F(\xi) = \frac{\theta(\xi)^{a-\gamma-1}(\xi - b)^{\gamma+N}g(\xi)}{\theta(\xi)^a - \theta(z)^a}.$$

Since $F(\xi)$ is analytic for ξ in the ring-shaped region between $C(p)$ and $C(|\theta(b)|)$, the two straight line segments of the contour C_ε cancel each other. Thus C_ε can be replaced by any contour $C(r)$ between $C(|\theta(b)|)$ and $C(p)$. Since z need no longer be on $C(|\theta(b)|)$ for $R_\varepsilon(N)$ to tend to zero in the proof of the previous theorem, the corollary is proved.

The generalized Taylor's series (1.2) involves a sum from $n = -\infty$ to ∞. Certain special cases of the sum over $n = 0$ to ∞ have appeared before [1, vol. 3, pp. 206–224]. In the following corollary we give a contour integral representation of this general sum.

COROLLARY 4.2. *With the hypothesis of Theorem* 4.1, *the formula*

$$\frac{\theta(z)^\gamma}{2\pi i}\int_C \frac{\theta(\xi)^{a-\gamma-1}\theta'(\xi)f(\xi)\,d\xi}{\theta(\xi)^a - \theta(z)^a} \tag{4.5}$$
$$= \sum_{n=0}^{\infty} \frac{D_{z-b}^{an+\gamma}[f(z)\theta'(z)((z - z_0)/\theta(z))^{an+\gamma+1}]|_{z=z_0}\,\theta(z)^{an+\gamma}}{\Gamma(an + \gamma + 1)}$$

is valid for all z inside the closed curve $C(|\theta(b)|)$. The contour of integration C starts at $\xi = b$, encloses the curve $C(|\theta(z)|)$ in the positive sense, and returns to $\xi = b$.

Proof. The corollary follows at once from the observation that the series (4.5) is generated by the integral (4.1) over the contour C_ε in the proof of Theorem 4.1.

We have seen that the contour integral definition of fractional differentiation (Definition 2.1) provides a convenient tool from which a proof of the generalized Taylor's series is constructed. In the next section this series is applied to the study of generating functions and other series expansions.

5. The discovery of generating functions and other examples. In this section we examine several examples of series which can be obtained from the generalized Taylor's series (1.2) by choosing specific functions for $f(z)$ and $\theta(z)$. A novel form of the binomial theorem is obtained as well as several generating functions for the special functions of mathematical physics. In fact, these examples reveal that the generalized Taylor's series is a very powerful tool for the discovery of generating functions when combined with the fractional derivative representations of the special functions such as those listed in Table 1.

In the examples which follow, ω and K are defined in the statement of Theorem 4.1, and the fractional derivatives encountered are computed with the aid of Table 1 and the extensive table in [2, vol. 2, pp. 185–200].

Example 1. Let $f(z) = z^p$, $\theta(z) = z - z_0$, and $b = 0$ in the generalized Taylor's series (1.2). We then obtain

$$\sum_{n=-\infty}^{\infty} \binom{p}{an+\gamma} t^{an+\gamma} = \begin{cases} a^{-1}(1+t)^p & \text{for } 0 < a \leqq 1, \\ a^{-1} \sum_{k\in K} \omega^{-\gamma k}(1+t\omega^k)^p & \text{for } 1 \leqq a \end{cases} \tag{5.1}$$

for $|t| = 1$, after making the substitution $t = (z - z_0)/z_0$. The special case in which $a = 1$ and $\gamma = 0$ is the familiar binomial expansion of $(1 + t)^p$. The case in which $a = 1$ and γ is arbitrary is an unusual form of the binomial theorem first stated by Riemann [12] and mentioned later by Heaviside [6, Chap. 7, 8], Watanabe [14] and Hardy [5]. The general case in which $0 < a \leqq 1$ and γ is arbitrary appears to be new.

Example 2. Let $f(z) = \sin\sqrt{z}$, $\theta(z) = z - z_0$, and $b = 0$ in (1.2). We then obtain the generating function

$$\sum_{n=-\infty}^{\infty} \frac{J_{1/2-an-\gamma}(x)t^{an+\gamma}}{\Gamma(an+\gamma+1)} = \begin{cases} a^{-1}\sqrt{2/(\pi x)}\sin\sqrt{x^2+2xt} & \text{for } 0 < a \leqq 1, \\ a^{-1}\sqrt{2/(\pi x)} \sum_{k\in K} \omega^{-\gamma k}\sin\sqrt{x^2+2xt\omega^k} & \text{for } 1 < a. \end{cases} \tag{5.2}$$

We have set $\sqrt{z_0} = x$ and $(z - z_0)/(2\sqrt{z_0}) = t$. The series (5.2) converges for $2|t| = |x|$. The special case of (5.2) in which $a = 1$ is well known [1, vol. 2, p. 100], while the general form for which $0 < a$ appears to be new.

Example 3. Let $f(z) = z^{(2\gamma-2\delta+1)/4}J_{1/2-\delta+\gamma}(\sqrt{z})$, $\theta(z) = z - z_0$, and $b = 0$ in (1.2). We then have

$$\sum_{n=-\infty}^{\infty} \frac{J_{1/2-\delta-an}(x)t^{an+\gamma}}{\Gamma(an+\gamma+1)} = \begin{cases} a^{-1}(1+2t/x)^{(2\gamma-2\delta+1)/4}J_{\gamma-\delta+1/2}(\sqrt{x^2+2tx}) & \text{for } 0 < a \leqq 1, \\ a^{-1}\sum_{k\in K}\omega^{-\gamma k}(1+2t\omega^k/x)^{(2\gamma-2\delta+1)/4}J_{\gamma-\delta+1/2}(\sqrt{x^2+2tx\omega^k}) & \text{for } 1 < a. \end{cases} \tag{5.3}$$

This series converges for $2|t| = |x|$. We have set $t = (z - z_0)/(2\sqrt{z_0})$, and $\sqrt{z_0} = x$. Equation (5.2) of the previous example is the special case of (5.3) in which $\delta = \gamma$. The special case of (5.3) in which $a = 1$ and $\gamma = 0$ is known as Lommel's formula [15, p. 140]. The general formula in which $0 < a$ appears to be new.

Example 4. In this example we take $\theta(z) = (z - 1)e^z$, so that we are using the Lagrange's expansion form of the generalized Taylor's series (1.2). Let $f(z) = e^{cz}z^{p-1}$ and $b = 0$. We then obtain

$$a^{-1}e^{cz}z^{p-1} = \sum_{n=-\infty}^{\infty} \binom{p}{an+\gamma} {}_1F_1(p+1; p - an - \gamma + 1;\ c - \gamma - an) \cdot ((z-1)e^z)^{an+\gamma}$$

for $0 < a \leqq 1$ and $|(z - 1)e^z| = 1$.

Example 5 (The discovery of generating functions). Examples 2, 3 and 4 above show that generating functions for the special functions of mathematical physics can readily be obtained from the generalized Taylor's series by a simple substitution of the fractional derivative representations for these special functions. Using this method, the author, who is not familiar with the clever manipulations of series so often encountered in this subject, was able to derive every generating function for the Bessel functions listed in the standard reference [1, vol. 3, Chap. 19]. These were obtained in a few hours from the fractional derivative representations of the Bessel functions.

Further examples of the use of the generalized Taylor's series in finding generating functions are given in [9].

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REFERENCES

[1] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, 3 vols., McGraw-Hill, New York, 1953, 1955.

[2] ———, *Tables of Integral Transforms*, 2 vols., McGraw-Hill, New York, 1954.

[3] A. ERDÉLYI, *An integral equation involving Legendre functions*, J. Soc. Indust. Appl. Math., 12 (1964), pp. 15–30.

[4] ———, *Axially symmetric potentials and fractional integration*, Ibid., 13 (1965), pp. 216–228.

[5] G. H. HARDY, *Riemann's form of Taylor's series*, J. London Math. Soc., 20 (1945), pp. 48–57.

[6] O. HEAVISIDE, *Electromagnetic Theory*, vol. 2, Dover, New York, 1950.

[7] T. P. HIGGINS, *The Rodrigues operator transform, table of generalized Rodrigues formulas*, Document D1-82-0493, Boeing Scientific Research Laboratories, Seattle, Washington, 1965.

[8] ———, *The use of fractional integral operators for solving nonhomogeneous differential equations*, Document D1-82-0677, Boeing Scientific Research Laboratories, Seattle, Washington, 1967.

[9] T. J. OSLER, *Leibniz rule, the chain rule and Taylor's theorem for fractional derivatives*, Doctoral thesis, New York University, New York, 1970.

[10] ———, *Leibniz rule for fractional derivatives generalized and an application to infinite series*, SIAM J. Appl. Math., 18 (1970), pp. 658–674.

[11] ———, *The fractional derivative of a composite function*, this Journal, 1 (1970), pp. 288–293.

[12] B. RIEMANN, *Versuch einer allgemeinen Auffasung der Integration und Differentiation*, The Collected Works of Bernhard Riemann, 2nd ed., H. Weber, ed., Dover, New York, 1953, pp. 353–366.

[13] M. Riesz, *L'integral de Riemann–Liouville et le problème de Cauchy*, Acta Math., 81 (1949), pp. 1–233.
[14] Y. Watanabe, *Zum Riemannschen binomischen Lehrsatz*, Proc. Phys. Math. Soc. Japan, 14 (1932), pp. 22–35.
[15] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1966.
[16] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1962.