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THEON'S LADDER FOR ANY ROOT

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Abstract

Theon's ladder is an ancient algorithm for calculating rational approximations for $\sqrt{2}$. It features two columns of integers, (called a ladder), in which the ratio of the two numbers in each row is an approximation to $\sqrt{2}$. It is remarkable for its simplicity. This algorithm can easily be generalized to find rational approximations to any square root. In this paper we show how Theon's original method is naturally generalized for the calculation of any root, $\sqrt[n]{c}$, where $1 < c$. In the generalization given here we require n columns of numbers as we generate rational approximations to an n th root. Several different recursion relations for the numbers that appear in the ladder are given, and a generating function for calculating the n th row of the ladder is found. Methods of increasing the rate of convergence are given, and a method of reducing the n-column ladder to a 2-column ladder is shown.

1. Introduction

Theon of Smyrna (circa 140 A. D.) described a remarkably simple way to calculate rational approximations to $\sqrt{2}$. It has become known as Theon's ladder and is shown below.

1	1
2	3
5	7
12	17
29	41
⋮	⋮

Each rung of the ladder contains two numbers. Call the left number on the *n* th rung x_n and the right number y_n. We see that $x_n = x_{n-1} + y_{n-1}$ and that $y_n = x_n + x_{n-1}$. So the next rung of the ladder has $x_6 =$ 29 + 41 = 70, and $y_6 =$ 70 + 29 = 99. The ratio of the two numbers on each rung, y_n / x_n gives us successively better approximations to $\sqrt{2}$.

1	1	$1/1 = 1.00000\cdots$
2	3	$3/2 = 1.50000\cdots$
5	7	$7/5 = 1.40000\cdots$
12	17	$17/12 = 1.41666\cdots$
29	41	$41/29 = 1.41379\cdots$
70	99	$99/70 = 1.41428\cdots$
169	239	$239/169 = 1.41420\cdots$

In [2] this ladder was generalized to obtain $\sqrt{c}$, where $1 \le c$, by using the recursion relations $x_n = x_{n-1} + y_{n-1}$, and $y_n = x_n + (c-1)x_{n-1}$. In this paper we show how to extend the ladder to obtain any root $\sqrt[N]{c}$. We will see that a three-column ladder produces cube roots, a four-column ladder produces fourth roots, etc. For example, to obtain $\sqrt[3]{2}$ we use the ladder

x_n	y_n	z_n	z_n / y_n
1	1	1	$1/1 = 1.00000000$
3	4	5	$5/4 = 1.25000000$
12	15	19	$19/15 = 1.26666667$
46	58	73	$73/58 = 1.25862069$
177	223	281	$281/223 = 1.260089686$

Here the elements of the ladder are calculated from $x_{n+1} = x_n + y_n + z_n$, $y_{n+1} = x_{n+1} + (c-1)x_n$ and $z_{n+1} = y_{n+1} + (c-1)y_n$, with $c = 2$. We see that the ratio z_n / y_n approaches $\sqrt[3]{c} = \sqrt[3]{2}$ =1.25992105.

The numerical calculation of roots using this ladder is extremely simple for students to understand. The theory behind the ladder requires only a knowledge of linear algebra.

2. The ladder for any root.

Let the elements of a general ladder with *N* columns be denoted in the following way:

$$\begin{array}{cccc} x_1(1) & x_1(2) & \cdots & x_1(N) \\ x_2(1) & x_2(2) & \cdots & x_2(N) \\ \cdots & & & \\ x_n(1) & x_n(2) & \cdots & x_n(N) \\ \cdots & & & \end{array}$$

Let the recursion relations for calculating the elements of rung $n+1$ from previously calculated elements on that rung and the previous rung be

$$x_{n+1}(1) = x_n(1) + x_n(2) + \cdots + x_n(N)$$

$$x_{n+1}(2) = x_{n+1}(1) + (c-1)x_n(1)$$

$$x_{n+1}(3) = x_{n+1}(2) + (c-1)x_n(2)$$

$\cdots$

$$x_{n+1}(N) = x_{n+1}(N-1) + (c-1)x_n(N-1)\,.$$

For the moment, we assume that the following two limits exist:

$$\lim_{n\to\infty} \frac{x_n(k)}{x_n(k-1)} = \alpha(k) \quad \text{for } k = 2,\ 3,\ \cdots,\ N\text{, and} \tag{2.1}$$

$$\lim_{n\to\infty} \frac{x_{n+1}(k)}{x_n(k)} = \gamma(k)\,,\ \text{ for } \ k = 1,\ 2,\ \cdots,\ N\,. \tag{2.2}$$

Later, in section 6, we will prove the existence of these limits.

Theorem 2.1: If the limits (2.1) and (2.2) exist, then $\gamma(1) = \gamma(2) = \gamma(3) = \cdots = \gamma(N)$ and $\alpha(2) = \alpha(3) = \alpha(4) = \cdots = \alpha(N)$.

Proof:

Starting with $x_{n+1}(k) = x_{n+1}(k-1) + (c-1)x_n(k-1)$, (valid for $k = 2,\ 3,\ \cdots,\ N$), we divide by $x_{n+1}(k-1)$ to get

$$\frac{x_{n+1}(k)}{x_{n+1}(k-1)} = 1 + (c-1)\frac{x_n(k-1)}{x_{n+1}(k-1)}\,.$$

Passing to the limit we get $\alpha(k) = 1 + \dfrac{c-1}{\gamma(k-1)}$, or

$$\alpha(k)\gamma(k-1) = \gamma(k-1) + c - 1\,,\ \text{ for } k = 2,\ 3,\ \cdots,\ N\,. \tag{2.3}$$

Next divide $x_{n+1}(k) = x_{n+1}(k-1) + (c-1)x_n(k-1)$ by $x_n(k)$ to get

$$\frac{x_{n+1}(k)}{x_n(k)} = \frac{x_{n+1}(k-1)}{x_n(k-1)}\frac{x_n(k-1)}{x_n(k)} + (c-1)\frac{x_n(k-1)}{x_n(k)}\,.$$

Passing to the limit we get $\gamma(k)=\dfrac{\gamma(k-1)}{\alpha(k)}+\dfrac{(c-1)}{\alpha(k)}$, or

$$\alpha(k)\gamma(k)=\gamma(k-1)+c-1, \text{ true for } k=2,3,\cdots,N. \tag{2.4}$$

From (2.3) and (2.4) it is clear that $\gamma(k)=\gamma(k-1)$ for $k=2,3,\cdots,N$. Thus $\gamma(1)=\gamma(2)=\gamma(3)=\cdots=\gamma(N)$, and we will call $\gamma(k)=\gamma$. Again, from (2.3) we see that $\alpha(k)=1+(c-1)/\gamma$ for $k=2,3,\cdots,N$. Thus $\alpha(2)=\alpha(3)=\alpha(4)=\cdots=\alpha(N)$, and we will call $\alpha(k)=\alpha$. This proves the theorem.

Now (2.3) can be written as

$$(\alpha-1)\gamma=c-1. \tag{2.5}$$

Notice that

$$\lim_{n\to\infty}\frac{x_n(4)}{x_n(2)}=\lim_{n\to\infty}\frac{x_n(4)}{x_n(3)}\frac{x_n(3)}{x_n(2)}=\alpha^2.$$

This simple observation can be extended easily to prove the following theorem.

Theorem 2.2: If the limits (2.1) and (2.2) exist, then

$$\lim_{n\to\infty}\frac{x_n(k+p)}{x_n(k)}=\alpha^p, \tag{2.6}$$

$$\lim_{n\to\infty}\frac{x_{n+q}(k)}{x_n(k)}=\gamma^q, \text{ and} \tag{2.7}$$

$$\lim_{n\to\infty}\frac{x_{n+q}(k+p)}{x_n(k)}=\alpha^p\gamma^q. \tag{2.8}$$

Here *k, p* and *q* are integers such that the quantities are defined.

Starting with $x_{n+1}(1)=x_n(1)+x_n(2)+\cdots+x_n(N)$, we divide by $x_n(1)$ to get

$$\frac{x_{n+1}(1)}{x_n(1)}=1+\frac{x_n(2)}{x_n(1)}+\cdots+\frac{x_n(N)}{x_n(1)}.$$

Passing to the limit and using Theorem 2.2 we get

$$\gamma = 1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1}.$$

Multiply this last relation by $\alpha - 1$ and use (2.5) to get

$$c - 1 = \left(1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1}\right)(\alpha - 1) = \alpha^N - 1.$$

Thus $\alpha^N = c$, and we have proved the following theorem.

Theorem 2.3: If the limits (2.1) and (2.2) exist, then $\alpha^N = c$. If in addition, c is positive, then $\lim_{n\to\infty} \frac{x_n(k+1)}{x_n(k)} = \sqrt[N]{c}$, where $k = 1, 2, \cdots, N-1$.

3. The multinomial connection

In [1], the remarkable binomial connection $\left(1 \pm \sqrt{c}\right)^n = x_n(2) \pm x_n(1)\sqrt{c}$ was demonstrated for the case where $N = 2$. In this section we show how this binomial relation is generalized for arbitrary N. First we prove the following lemma which shows how to calculate the elements on rung $n+1$ using only the elements of rung n.

Lemma 3.1: The following recursion relations are true:

$$x_{n+1}(1) = x_n(1) + x_n(2) + x_n(3) + x_n(4) + \cdots + x_n(N)$$

$$x_{n+1}(2) = x_n(1)c + x_n(2) + x_n(3) + x_n(4) + \cdots + x_n(N)$$

$$x_{n+1}(3) = x_n(1)c + x_n(2)c + x_n(3) + x_n(4) + \cdots + x_n(N)$$

$$x_{n+1}(4) = x_n(1)c + x_n(2)c + x_n(3)c + x_n(4) + \cdots + x_n(N)$$

$$\cdots.$$

$$x_{n+1}(N) = x_n(1)c + x_n(2)c + x_n(3)c + \cdots + x_n(N-1)c + x_n(N).$$

Proof:

The first relation, for $x_{n+1}(1)$, is the relation we have used to define it. The second relation follows from

$x_{n+1}(2) = x_{n+1}(1) + (c-1)x_n(1) = x_n(1) + x_n(2) + \cdots + x_n(N) + (c-1)x_n(1)$

$= x_n(1)c + x_n(2) + x_n(3) + x_n(4) + \cdots + x_n(N)$. The remaining relations follow in the same way.

Let $\omega = \exp(2p\pi i / N)$, where p is any integer, so $\omega^N = 1$. We will prove the following:

Theorem 3.1: If $x_1(1) = x_1(2) = x_1(3) = \cdots = x_1(N) = 1$, then

$$\begin{aligned}&\left(x_n(N) + x_n(N-1)\omega\, c^{1/N} + x_n(N-2)\omega^2 c^{2/N} + \cdots + x_n(1)\omega^{N-1}c^{(N-1)/N}\right) = \\ &\left(1 + \omega\, c^{1/N} + \omega^2 c^{2/N} + \cdots + \omega^{N-1}c^{(N-1)/N}\right)^n.\end{aligned} \tag{3.1}$$

Proof:

The proof is by induction on *n*. The case where $n = 1$ is clear. Assume (3.1) is true for rung *n*. Then $\left(1 + \omega\, c^{1/N} + \omega^2 c^{2/N} + \cdots + \omega^{N-1}c^{(N-1)/N}\right)^{n+1}$ is equal to the product of

$\left(x_n(N) + x_n(N-1)\omega\, c^{1/N} + x_n(N-2)\omega^2 c^{2/N} + \cdots + x_n(1)\omega^{N-1}c^{(N-1)/N}\right)$ and

$\left(1 + \omega\, c^{1/N} + \omega^2 c^{2/N} + \cdots + \omega^{N-1}c^{(N-1)/N}\right)$.

This product is

$$\begin{aligned}
&x_n(N) + \quad x_n(N-1)wc^{\frac{1}{N}} + \quad x_n(N-2)w^2c^{\frac{2}{N}} + \ldots + x_n(1)w^{N-1}c^{\frac{N-1}{N}} + \\
&x_n(1)c + \quad x_n(N)wc^{\frac{1}{N}} + \quad x_n(N-1)w^2c^{\frac{2}{N}} + \ldots + \quad x_n(2)w^{N-1}c^{\frac{N-1}{N}} + \\
&x_n(2)c + \quad x_n(1)cwc^{\frac{1}{N}} + \quad x_n(N)w^2c^{\frac{2}{N}} + \ldots + \quad x_n(3)w^{N-1}c^{\frac{N-1}{N}} + \\
&\qquad \ldots +
\end{aligned}$$

$$x_n\left(N-1\right)c+x_n\left(N-2\right)cwc^{\frac{1}{N}}+x_n\left(N-3\right)cw^2c^{\frac{2}{N}}+...+x_n(N)w^{N-1}c^{\frac{N-1}{N}}$$

Summing by columns we get

$$\left(x_n(N)+x_n(N-1)c+x_n(N-2)c+x_n(N-3)c+\cdots+x_n(1)c\right)+$$

$$\left(x_n(N)+x_n(N-1)+x_n(N-2)c+x_n(N-3)c+\cdots+x_n(1)c\right)\omega c^{1/N}+$$

$$\left(x_n(N)+x_n(N-1)+x_n(N-2)+x_n(N-3)c+\cdots+x_n(1)c\right)\omega^2c^{2/N}+$$

$$\cdots+$$

$$\left(x_n(N)+x_n(N-1)+x_n(N-2)+x_n(N-3)+\cdots+x_n(1)\right)\omega^{N-1}c^{(N-1)/N}$$

The result now follows at once from *Lemma 3.1*.

Notice that (3.1) states that the expression

$\left(1+\omega\ c^{1/N}+\omega^2c^{2/N}+\cdots+\omega^{N-1}c^{(N-1)/N}\right)^n$ is a generating function the elements of the n th rung.

4. The matrix connection

The results of *Lemma 3.1* show that we should consider introducing the following vectors and matrices. Let $\mathbf{x_n}=\left(x_n(1),x_n(2),x_n(3),\cdots,x_n(N)\right)$. We will use vectors as rows and as columns as the need arises without changing the vector notation. Let the important matrix implied by *Lemma 3.1* with N rows and columns be denoted by

$$\mathbf{M}(N)=\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ c & 1 & 1 & 1 & & 1 \\ c & c & 1 & 1 & & 1 \\ c & c & c & 1 & & 1 \\ \cdots & & & & & \\ c & c & c & \cdots & c & 1 \end{pmatrix}.$$

From lemma 3.1 we then have

$$\mathbf{x_{n+1}} = \mathbf{M}(N)\,\mathbf{x_n}\,. \tag{4.1}$$

First we prove a lemma which will show that $\mathbf{M}(N)$ is not singular when $c \neq 1$.

Lemma 4.1: The determinant of the matrix $\mathbf{M}(N)$ is given by $\det(\mathbf{M}(N)) = (1-c)^{N-1}$.

Proof: In $\mathbf{M}(N)$, subtract column 1 from each column. The first row becomes 1 0 0 0 … 0. Expand by the first row. The first $(n-1)\times(n-1)$ minor has $(1-c)$ along its major diagonal and zeroes everywhere beneath. Thus $\det(\mathbf{M}(N)) = (1-c)^{N-1}$.

Theorem 4.1: Let $1 < c$ and $\omega = e^{2\pi i/N}$. Then the eigenvalues and corresponding eigenvectors for the matrix $\mathbf{M}(N)$ are

$$\lambda_k = 1 + \omega^{k-1}c^{1/N} + \omega^{2(k-1)}c^{2/N} + \cdots + \omega^{(N-1)(k-1)}c^{(N-1)/N} \text{ and} \tag{4.2}$$

$$\mathbf{v_k} = \left(1,\, \omega^{k-1}c^{1/N},\, \omega^{2(k-1)}c^{2/N},\, \cdots,\, \omega^{(N-1)(k-1)}c^{(N-1)/N}\right), \tag{4.3}$$

for $k = 1, 2, 3, \cdots, N$. Also, the eigenvectors are linearly independent.

Proof:

Direct substitution of the eigenvalues and eigenvectors shows that the relations $\mathbf{M}(N)\mathbf{v_k} = \lambda_k \mathbf{v_k}$ are true for $k = 1, 2, 3, \cdots, N$.

It remains to show that the eigenvectors are linearly independent. We must show that if

$$\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_N\mathbf{v_N} = 0\,,$$

then numbers $\alpha_1, \alpha_2, \cdots, \alpha_N$, are all zero. This means that we examine the system of *N* equations

$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_N = 0$$

$$\alpha_1 + \alpha_2\omega + \alpha_3\omega^2 + \alpha_4\omega^4 + \cdots + \alpha_N\omega^{N-1} = 0$$

$$\alpha_1 + \alpha_2\omega^2 + \alpha_3\omega^4 + \alpha_4\omega^6 + \cdots + \alpha_N\omega^{2(N-1)} = 0$$

$$\alpha_1 + \alpha_2\omega^3 + \alpha_3\omega^6 + \alpha_4\omega^9 + \cdots + \alpha_N\omega^{3(N-1)} = 0$$

$$\cdots$$

$$\alpha_1 + \alpha_2\omega^{N-1} + \alpha_3\omega^{2(n-1)} + \alpha_4\omega^{3(n-1}{}_{-} + \cdots + \alpha_N\omega^{(N-1)(N-1)} = 0,$$

and show that the only solution is $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$. The equation $\alpha_1 + \alpha_2 z + \alpha_3 z^2 + \alpha_4 z^4 + \cdots + \alpha_N z^{N-1} = 0$ can have at most N-1 distinct roots if all the coefficients are not zero. But the above equations show that we have N roots $z = 1,\ \omega,\ \omega^2,\ \cdots,\ \omega^{N-1}$. Thus all the coefficients $\alpha_k = 0$ for k = 1, 2, 3, …, N, and the theorem is proved.

The first rung of the ladder is given in our vector notation by $\mathbf{x_1}$ and by the previous theorem, can be expanded in terms of the eigenvectors as

$$\mathbf{x_1} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_N\mathbf{v_N},$$

where $a_1,\ a_2,\ a_3,\ \cdots,\ a_N$ are appropriately chosen constants. Then

$$\begin{aligned}\mathbf{x_{n+1}} &= \mathbf{M}^n(N)\mathbf{x_1} \\ &= \mathbf{M}^n(N)\left(a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_N\mathbf{v_N}\right) \\ &= a_1\lambda_1^n\mathbf{v_1} + a_2\lambda_2^n\mathbf{v_2} + \cdots + a_N\lambda_N^n\mathbf{v_N}.\end{aligned}$$

Writing out the components of $\mathbf{x_{n+1}}$ from the above relation and using (4.3) we can calculate the numbers on any rung from the eigenvalues.

$$x_{n+1}(1) = a_1\lambda_1^n + a_2\lambda_2^n + a_3\lambda_3^n + \cdots + a_N\lambda_N^n, \tag{4.5}$$

$$x_{n+1}(2) = \left(a_1\lambda_1^n + a_2\lambda_2^n\omega + a_3\lambda_3^n\omega^2 + \cdots + a_N\lambda_N^n\omega^{N-1}\right)c^{1/N}$$

$$x_{n+1}(3) = \left(a_1\lambda_1^n + a_2\lambda_2^n\omega^2 + a_3\lambda_3^n\omega^{2\cdot 2}c^{2/N} + \cdots + a_N\lambda_N^n\omega^{2(N-1)}\right)c^{2/N}$$

…

$$x_{n+1}(N) = \left(a_1\lambda_1^n + a_2\lambda_2^n\omega^{N-1} + a_3\lambda_3^n\omega^{(N-1)\cdot 2} + \cdots + a_N\lambda_N^n\omega^{(N-1)(N-1)}\right)c^{(N-1)/N}.$$

5. Proof that the limits exist

It is now possible to prove that the limits considered above exist.

Theorem 5.1: Let the first rung of the ladder be $\mathbf{x_1} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_N\mathbf{v_N}$, and let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_N$ be the eigenvectors given by (4.3). If $1 < c$ and $a_1 \neq 0$, then the limits

$$\alpha(k) = \lim_{n\to\infty}\frac{x_n(k)}{x_n(k-1)} \quad \text{for } k = 2, 3, \cdots, N\text{, and} \tag{5.1}$$

$$\gamma(k) = \lim_{n\to\infty}\frac{x_{n+1}(k)}{x_n(k)}, \text{ for } k = 1, 2, \cdots, N \tag{5.2}$$

exist.

Proof:

From (4.5) we have

$$\frac{x_{n+1}(k)}{x_n(k)} = \frac{a_1\lambda_1^n + a_2\lambda_2^n\omega^{k-1}c^{1/N} + a_3\lambda_3^n\omega^{(k-1)\cdot 2}c^{2/N} + \cdots + a_N\lambda_N^n\omega^{(k-1)(N-1)}c^{(N-1)/N}}{a_1\lambda_1^{n-1} + a_2\lambda_2^{n-1}\omega^{k-1}c^{1/N} + a_3\lambda_3^{n-1}\omega^{(k-1)\cdot 2}c^{2/N} + \cdots + a_N\lambda_N^{n-1}\omega^{(k-1)(N-1)}c^{(N-1)/N}}$$

Dividing numerator and denominator by λ_1^{n-1} we have

$$\frac{x_{n+1}(k)}{x_n(k)} = \frac{a_1\lambda_1 + a_2\lambda_2\frac{\lambda_2^{n-1}}{\lambda_1^{n-1}}\omega^{k-1}c^{1/N} + a_3\lambda_3\frac{\lambda_3^{n-1}}{\lambda_1^{n-1}}\omega^{(k-1)\cdot 2}c^{2/N} + \cdots + a_N\lambda_N\frac{\lambda_N^{n-1}}{\lambda_1^{n-1}}\omega^{(k-1)(N-1)}c^{(N-1)/N}}{a_1 + a_2\frac{\lambda_2^{n-1}}{\lambda_1^{n-1}}\omega^{k-1}c^{1/N} + a_3\frac{\lambda_3^{n-1}}{\lambda_1^{n-1}}\omega^{(k-1)\cdot 2}c^{2/N} + \cdots + a_N\frac{\lambda_N^{n-1}}{\lambda_1^{n-1}}\omega^{(k-1)(N-1)}c^{(N-1)/N}}$$

.

Since $\left|\frac{\lambda_h}{\lambda_1}\right| < 1$ for $h = 2, 3, \cdots, N$, we see that $\lim_{n\to\infty}\frac{\lambda_h^n}{\lambda_1^n} = 0$. Thus $\lim_{n\to\infty}\frac{x_{n+1}(k)}{x_n(k)} = \lambda_1$, for $k = 1, 2, \cdots, N$, and we have shown that the limits in (5.2) exist. Notice that

$\lambda_1 = 1 + c^{1/N} + c^{2/N} + \cdots + c^{(N-1)/N} = \dfrac{c-1}{c^{1/N}-1}$, which agrees with Theorem 2.3 and (2.5).

From (2.3) it follows that the limits (5.1) exist and the theorem is proved.

6. Skipping rungs of the ladder

The convergence of our ladder is slow, and we can improve the convergence in a simple way. Let $\omega = 1$ in (3.1) and square to obtain

$$\begin{aligned}
&\left(x_n(N) + x_n(N-1)c^{1/N} + x_n(N-2)c^{2/N} + \cdots + x_n(1)c^{(N-1)/N}\right)^2 = \\
&\left(1 + c^{1/N} + c^{2/N} + \cdots + c^{(N-1)/N}\right)^{2n} = \\
&\left(x_{2n}(N) + x_{2n}(N-1)c^{1/N} + x_{2n}(N-2)c^{2/N} + \cdots + x_{2n}(1)c^{(N-1)/N}\right).
\end{aligned} \tag{6.1}$$

It is instructive to study a few special cases of the above relations. If $N = 2$ we get

$$\left(x_n(2) + x_n(1)c^{1/2}\right)^2 = \left(x_n(2)^2 + x_n(1)^2 c\right) + 2x_n(1)x_n(2)c^{1/2}$$

$$= x_{2n}(2) + x_{2n}(1)c^{1/2}.$$

Thus we have

$$x_{2n}(1) = 2x_n(1)x_n(2), \text{ and} \tag{6.2}$$

$$x_{2n}(2) = x_n(2)^2 + x_n(1)^2 c.$$

Again, if $N = 3$ in (6.2) we get

$$\left(x_n(3) + x_n(2)c^{1/3} + x_n(1)c^{2/3}\right)^2 =$$

$$\left(x_n^2(3) + 2x_n(1)x_n(2)c\right) + \left(x_n^2(1)c + 2x_n(2)x_n(3)\right)c^{1/3} + \left(x_n^2(2) + 2x_n(1)x_n(3)\right)c^{2/3} =$$

$$x_{2n}(3) + x_{2n}(2)c^{1/3} + x_{2n}(1)c^{2/3}.$$

Thus we conclude that

$$x_{2n}(1) = x_n^2(2) + 2x_n(1)x_n(3), \tag{6.3}$$

$$x_{2n}(2) = x_n^2(1)c + 2x_n(2)x_n(3), \text{ and}$$

$$x_{2n}(3) = x_n^2(3) + 2x_n(1)x_n(2)c\ .$$

For general *N* and $1 \le m \le N$, we have

$$x_{2n}(m) = c\sum_{k=1}^{m-1} x_n(k)x_n(m-k) + \sum_{k=m}^{N} x_n(k)x_n(N+m-k)\ . \qquad (6.4)$$

Using (6.4), the elements in rung $2n$ can be calculated from the elements on rung *n*. The resulting limit of the ratio $\frac{x_{2n}(m+1)}{x_{2n}(m)}$ converges quadratically to $c^{1/N}$ for $1 < c$. (We will not demonstrate this quadratic convergence.)

7. *N* columns reduced to two columns

We will now show how to reduce our *N* column ladder to a two-column ladder. We begin by considering the case where *N* = 3. The fundamental recursion relations are

$$x_{n+1}(1) = x_n(1) + x_n(2) + x_n(3)\ , \qquad (7.1)$$

$$x_{n+1}(2) = x_{n+1}(1) + (c-1)x_n(1)\text{, and} \qquad (7.2)$$

$$x_{n+1}(3) = x_{n+1}(2) + (c-1)x_n(2)\ . \qquad (7.3)$$

Using (7.2) to eliminate $x_n(2)$ and $x_{n+1}(2)$ from (7.3) we get

$$x_{n+1}(3) = x_{n+1}(1) + (c-1)x_n(1) + (c-1)\left(x_n(1) + (c-1)x_{n-1}(1)\right)$$

$$= x_{n+1}(1) + 2(c-1)x_n(1) + (c-1)^2 x_{n-1}(1)\ .$$

Using this last result and (7.2) with (7.1) we get

$$x_{n+1}(1) = 3x_n(1) + 3(c-1)x_{n-1}(1) + (c-1)^2 x_{n-2}(1)\ .$$

This last relation combined with (7.2) allows us to calculate a two-column ladder to estimate $c^{1/3}$. The resulting ladder contains the first two columns of the original three-column ladder for $c^{1/3}$.

In the same way, we can reduce our general *N* column ladder to a two column ladder using

$$x_{n+1}(1) = \sum_{m=1}^{N} \binom{N}{m} (c-1)^{m-1} x_{n-m+1}(1) \text{, and} \tag{7.4}$$

$$x_{n+1}(2) = x_{n+1}(1) + (c-1)x_n(1) .$$

Equation (7.4) is not only true for the first column, but it is true for all columns as shown in the following theorem.

Theorem 7.1: For $1 \le k \le N$ we have

$$x_{n+1}(k) = \sum_{m=1}^{N} \binom{N}{m} (c-1)^{m-1} x_{n-m+1}(k) . \tag{7.5}$$

Proof: Our proof is by induction on *k*. By (7.4), the result is true for $k=1$. Assume (7.5) is true for a specific *k,* with $k < N$. We investigate $x_{n+1}(k+1)$. We have

$$x_{n+1}(k+1) = x_{n+1}(k) + (c-1)x_n(k) .$$

From the induction hypothesis we have

$$x_{n+1}(k+1) = \sum_{m=1}^{N} \binom{N}{m} (c-1)^{m-1} x_{n-m+1}(k) + (c-1)\sum_{m=1}^{N} \binom{N}{m} (c-1)^{m-1} x_{n-m}(k)$$

$$= \sum_{m=1}^{N} \binom{N}{m} (c-1)^{m-1} \left(x_{n-m+1}(k) + (c-1)x_{n-m}(k) \right)$$

$$= \sum_{m=1}^{N} \binom{N}{m} (c-1)^{m-1} x_{n-m+1}(k+1) .$$

The theorem is proved.

8. Final remarks

In [1], Eisenberg gave another discussion of an algorithm for the square root identical to the one used in Theon's ladder. In his excellent book [4], Young describes

Theon's ladder for the square root and gives several interesting problems related to it on . pages 10-11, 38-41, and 96-97. Gould gave several interesting algorithms related to this work in [3].

References

[1] Eisenberg, Theodore, *On an unknown algorithm for computing square roots*, International Journal of Mathematical Education in Science and Technology, 34(2003), pp. 153-158.

[2] Giberson, S. and Osler, T. J., *Extending Theon's ladder to any square root*, to appear in The College Mathematics Journal.

[3] Gould, H. W., *An Iterative Approximation for Finding the N-th Root of a Number*, Mathematics Magazine, Vol. 33, No. 2. (Nov. - Dec., 1959), pp. 61-69.

[4] Young, Robert M., *Excursions in Calculus, An Interplay of the Continuous and the Discrete*, The Mathematical Association of America, 1992.