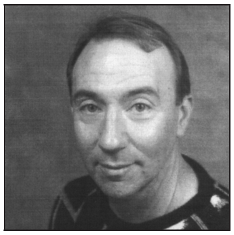
A Tale of Two Series

Thomas J. Osler and Marcus Wright



Tom Osler (osler@rowan.edu) is professor of mathematics at Rowan University. He received his Ph.D. from the Courant Institute at New York University in 1970 and is the author of thirty-eight mathematical papers. In addition to teaching university mathematics, Tom has a passion for long distance running. He has been competing for the past forty-seven years. Included in his over 1600 races are wins in three national championships in the late sixties at distances from 25 kilometers to 50 miles. He is the author of two running books.

Marcus Wright (wright@rowan.edu) received his A.B. from Harvard University and his Ph.D. from Stanford University. Gradually moving east, he taught at Rice, the University of Kentucky, and Rutgers before settling at Rowan. His research interests are in the areas of complex differential geometry, iteration of rational functions, and numerical methods.

Introduction

They are the best of series, they are the worst of series. One series converges for all z, but often fails to provide meaningful numerical values. One series diverges for all z yet often gives excellent numerical values. What the Dickens is going on?

The two series with the perplexing properties arise from an important problem, finding the area under the normal distribution curve $\frac{e^{-z^2/2}}{\sqrt{2\pi}}$ from z to infinity. Our first series will be obtained using integration by parts in the next section. It is

$$P(z) \sim \frac{e^{-z^2/2}}{\sqrt{2\pi}\,z} \sum_{n=0}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{z^{2n}}. \tag{1.1}$$

(The symbol $\sim$ means that the series on the right is asymptotic to the function on the left. This will be explained later.) The second series (again found using integration by parts) is

$$P(z) = \frac{1}{2} - \frac{e^{-z^2/2}}{\sqrt{2\pi}\,z} \sum_{n=1}^{\infty} \frac{z^{2n}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}. \tag{1.2}$$

The ratio test easily shows that the series (1.1) is divergent for all values of z, while the second series (1.2) converges for all z. Surely (1.2) is the preferred series for numerical evaluation? Not always!

For very small values of z, the convergent series is certainly preferable, but for values of z greater than 5 or so, a few terms of the divergent series (1.1) will give very accurate numerical values of $P(z)$ while the convergent series might need hundreds of terms and might even fail due to loss of significant digits. For large values of z, the divergent series is much better for numerical calculations if used correctly.

When students first study series, they often imagine that convergent series are "good" series and divergent series are "bad". This is often true, but not always as this paper will show. The series (1.1) is known as an asymptotic series. Asymptotic series diverge, yet their first few terms can be used to give very accurate numerical values for the functions they represent. Using additional terms will not increase the accuracy of the approximation, but decrease it. Asymptotic series might be studied in a graduate course, but the series (1.1) is so simple that it can be understood by any calculus student who knows integration by parts. We hope that this example will provide a stimulating hour for undergraduate students in calculus, numerical analysis, and real analysis, as well as professionals unfamiliar with this subject.

Deriving the divergent series

We will now use integration by parts to derive the series (1.1) for the area under the normal probability curve $P(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-t^2/2}\, dt$. Letting $u = 1/t$ and $dv = te^{-t^2/2}$, we get

$$\int_z^\infty e^{-t^2/2}\, dt = \int_z^\infty \frac{d(e^{-t^2/2})}{t} = \frac{e^{-z^2/2}}{z} - \int_z^\infty \frac{e^{-t^2/2}}{t^2}\, dt. \tag{2.1}$$

Integrating by parts again, using $u = 1/t^3$ and $dv = te^{-t^2/2}$, (2.1) becomes

$$\frac{e^{-z^2/2}}{z} + \int_z^\infty \frac{de^{-t^2/2}}{t^3} = \frac{e^{-z^2/2}}{z} - \frac{e^{-z^2/2}}{z^3} + 3\int_z^\infty \frac{e^{-t^2/2}}{t^4}\, dt.$$

Continuing in this way we obtain

$$P(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-t^2/2}\, dt = \frac{e^{-z^2/2}}{\sqrt{2\pi}\, z} \sum_{n=0}^{N} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{z^{2n}} + R_D(z, N) \tag{2.2}$$

where

$$R_D(z, N) = \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2N+1)}{\sqrt{2\pi}} \int_z^\infty \frac{e^{-t^2/2}}{t^{2(N+1)}}\, dt.$$

We now find a bound for this remainder. Assume that z is a real positive number. Multiply the numerator and denominator of the integrand by t to get

$$R_D(z, N) = \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2N+1)}{\sqrt{2\pi}} \int_z^\infty \frac{te^{-t^2/2}}{t^{2N+3}}\, dt.$$

The largest value that the denominator t^{2N+3} contributes to the integral occurs when $t = z$. Thus we can bound the remainder by

$$|R_D(z, N)| < \frac{1 \cdot 3 \cdot 5 \cdots (2N+1)}{\sqrt{2\pi}\, z^{2N+3}} \int_z^\infty te^{-t^2/2}\, dt.$$

This last integral is easily evaluated to give

$$|R_D(z,N)| < \frac{1\cdot 3\cdot 5\cdots(2N+1)}{\sqrt{2\pi}\,z^{2N+3}}e^{-z^2/2}, \tag{2.3}$$

so the remainder is less than the first term neglected.

Since $1\cdot 3\cdot 5\cdots(2n-1) = \frac{(2n)!}{2^n n!}$, we can rewrite (2.1) as

$$P(z) = \frac{1}{\sqrt{2\pi}}\int_z^\infty e^{-t^2/2}\,dt = \frac{e^{-z^2/2}}{\sqrt{2\pi}\,z}\sum_{n=0}^{N}\frac{(-1)^n(2n)!}{2^n n!\,z^{2n}} + R_D(z,N). \tag{2.4}$$

Using the ratio test it is easy to see that the partial sums in (2.1) will diverge if N is allowed to continue to infinity. However, using a few terms of the series will allow us to estimate the area under the normal curve quite well for large values of z. This will be made clear in section 4 by examples. Because we have the bound on the remainder (2.2), we will be able to predict the accuracy of such approximations.

The convergent series

We will now obtain a convergent series for the probability integral

$$P(z) = \frac{1}{\sqrt{2\pi}}\int_z^\infty e^{-t^2/2}\,dt.$$

Since $\frac{e^{-z^2/2}}{\sqrt{2\pi}}$ is the normal probability distribution, the area under it from $-\infty$ to ∞ is 1. Therefore the area from 0 to ∞ is $1/2$, and we can rewrite the integral as

$$P(z) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\int_0^z e^{-t^2/2}\,dt.$$

We have, integrating by parts, using $u = e^{-t^2/2}$ and $dv = 1\,dt$,

$$\begin{aligned}
P(z) &= \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\int_0^z e^{-t^2/2}\,dt = \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\left[ze^{-z^2/2} + \int_0^z t^2 e^{-t^2/2}\,dt\right] \\
&= \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\left[ze^{-z^2/2} + \int_0^z e^{-t^2/2} d\left(\frac{t^3}{3}\right)\right] \\
&= \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\left[ze^{-z^2/2} + \frac{z^3 e^{-z^2/2}}{3} + \frac{1}{3}\int_0^z e^{-t^2/2} d\left(\frac{t^5}{5}\right)\right].
\end{aligned}$$

Continuing in this way we get

$$P(z) = \frac{1}{2} - \frac{e^{-z^2/2}}{\sqrt{2\pi}\,z}\sum_{n=1}^{N}\frac{z^{2n}}{1\cdot 3\cdot 5\cdots(2n-1)} + R_C(z,N), \tag{3.1}$$

where

$$R_C(z,N) = \frac{1}{1\cdot 3\cdot 5\cdots(2N+1)}\int_0^z t^{2N}e^{-t^2/2}dt < \frac{z^{2N}}{1\cdot 3\cdot 5\cdots(2N+1)}.$$

It is easy to see that this remainder goes to zero as $N \to \infty$ and so this series converges to $P(z)$. As before, using (2.3) we can rewrite this sum as

$$P(z) = \frac{1}{2} - \frac{e^{-z^2/2} z}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{2^n n!\, z^{2n}}{(2n)!}. \tag{3.2}$$

Comparing the two series

We now call the Nth partial sum of our divergent series (2.1) $Pd(N, z)$ and write

$$Pd(z, N) = \frac{e^{-z^2/2}}{\sqrt{2\pi} z} \sum_{n=0}^{N} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{z^{2n}}. \tag{4.1}$$

For our convergent series (3.2) we will call the Nth partial sum $Pc(N, z)$ and write

$$Pc(N, z) = \frac{1}{2} - \frac{e^{-z^2/2}}{\sqrt{2\pi} z} \sum_{n=1}^{N} \frac{z^{2n}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}. \tag{4.2}$$

We will refer to the absolute value of the general term in our divergent series as $td(n, z)$:

$$td(z, n) = \frac{e^{-z^2/2}}{\sqrt{2\pi} z} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{z^{2n}}. \tag{4.3}$$

Also, we refer to the absolute value of the general term in our convergent series (4.2) as $tc(n, z)$:

$$tc(z, n) = \frac{e^{-z^2/2}}{\sqrt{2\pi} z} \frac{z^{2n}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}. \tag{4.4}$$

For fixed z, the product $tz(n, z) \cdot tc(n, z)$ is constant. Hence when our upper bound (i.e., $td(n, z)$) on the error in using $Pd(n, z)$ to approximate P(z) has its minimum, our lower bound on the error in using $Pc(n, z)$ (i.e., $tc(n, z)$) will have its maximum.

Table 4.1 was created with the help of the program Mathematica. In the second column are values of the probability integral $P(z)$ computed to an accuracy of 20

Table 4.1. Comparing series for various z

z	$P(z)$	$Pc(50, z)$	$Pd(5, z)$
1	0.1586552	0.1586552	22.503277
2	0.0227501	0.0227501	0.0300535
3	0.0013498	0.0013498	0.0013610
4	0.0000316	0.0000316	0.0000316
5	2.8665157×10^{-7}	2.8665157×10^{-7}	2.8667187×10^{-7}
6	$9.8658764 \times 10^{-10}$	1.1189081×10^{-9}	$9.8659988 \times 10^{-10}$
7	$1.2798125 \times 10^{-12}$	1.4128472	$1.2798161 \times 10^{-12}$
8	$6.2209605 \times 10^{-16}$	0.0007573	$6.2209653 \times 10^{-16}$
9	$1.1285884 \times 10^{-19}$	0.0357440	$1.1285886 \times 10^{-19}$
10	$7.6198530 \times 10^{-24}$	0.2452877	$7.6198536 \times 10^{-24}$

significant digits. In the third column we have the result of using the convergent series (4.2) summed to 50 terms and in the last column the divergent series (4.1) summed to only 5 terms.

From Table 4.1, we notice that

(1) For values of $z \le 5$, 50 terms of the convergent series give better results than 5 terms of the divergent series.

(2) For values of $z > 5$ the divergent series is superior. For example, with only five terms of this series, $P(10)$ can be computed to an accuracy of 7 significant digits.

Table 4.2. Convergent series for $P(10)$

n	$tc(n, 10)$	$Pc(n, 10)$
0	7.694599×10^{-24}	0.500000
1	7.694599×10^{-22}	0.500000
2	2.564866×10^{-20}	0.500000
⋮	⋮	⋮
49	0.027950	0.273521
50	0.028233	0.245288
51	0.027953	0.217334
⋮	⋮	⋮
175	3.352748×10^{-43}	7.619853×10^{-24}
176	9.551989×10^{-44}	7.619853×10^{-24}
177	2.705946×10^{-44}	7.619853×10^{-24}

Next we examine how the convergent series (4.2) behaves in Table 4.2. Here we see the result of setting $z = 10$. The value of $tc(n, z)$ gradually increases until at $n = 50$ it has the maximum value $tc(50, 10) = 0.028233$. As n continues to increase we see this term decreases to 2.705946×10^{-44} when $n = 177$. This behavior will be explained in the next section. The value of the partial sum given by Mathematica when $n = 177$ is $Pc(177, 10) = 7.619853024160526066 \times 10^{-24}$. We list these three numbers so that the digits line up:

$$tc(50, 10) = 0.02823\ 29963\ 25476\ 24742\ 53651\ 79423\ 15660\ 57781\ 31228$$

$$|\longleftarrow \quad 42 \textit{ significant digits required during calculation } \longrightarrow|$$

$$tc(177, 10) = 0.00000\ 00000\ 00000\ 00000\ 00000\ 00000\ 00000\ 00000\ 00027$$

$$|\leftarrow 20 \textit{ digit correct result } \rightarrow|$$

$$Pc(177, 10) = 0.00000\ 00000\ 00000\ 00000\ 00076\ 19853\ 02416\ 05260\ 65983$$

Now we can see that the computer must keep about 42 significant digits in each term to obtain only 20 significant digits in this last approximation to $P(10)$. This is an expensive loss of significant digits.

How well can the divergent series approximate $P(10)$? From (2.3), the error in using this series is less than the first term neglected. Table 4.2 shows that the maximum value of $tc(n, 10)$ occurs when $n = 50$ and hence the minimum value of $td(n, 10)$ is at $n = 50$. Since the value of this term is 2.1×10^{-45}, a limit is placed on how accurately we can use this series to estimate $P(10)$. We can have at most 44 correct digits after the decimal point. Since the sum of the divergent series up to the 49th term is $Pd(49, 10) = 7.6198530 \times 10^{-24}$, we see that the first significant figure begins in the 24th decimal place. With a limit of 44 correct decimal places, this means that no more than 21 significant digits can ever be calculated from the divergent series expansion when $z = 10$. While $P(z)$ is more easily calculated from our divergent series for large z, there is a limit to the number of significant digits that can be guaranteed.

Best and worst of the series

In the previous section, we saw an example in which the maximum (worst) term in the convergent series expansion for $P(z)$ (given by (4.4)) occurred when $n = z^2/2$. In this section we will indicate why this is so. Our derivation is not rigorous, but we will show that for fixed z

$$tc(n, z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \frac{2^n n!\, z^{2n+1}}{(2n)!} \tag{5.1}$$

attains a maximum when n is about $z^2/2$.

Since z is fixed, we need only find the maximum of

$$t(n) = \frac{2^n n!\, z^{2n+1}}{(2n)!} \tag{5.2}$$

as n varies. It would be nice to set the derivative of (5.2) with respect to n equal to zero in the usual calculus fashion, but n is not a continuous variable and the factorials cannot be differentiated easily. However, for large x we know that the factorials can be approximated by Stirling's expansion,

$$x! \approx \sqrt{2\pi x}\, x^x e^{-x}. \tag{5.3}$$

Substituting (5.3) for the factorials in (5.2) we get

$$t(n) \approx \frac{2^n \sqrt{2\pi n}\, n^n e^{-n} z^{2n+1}}{\sqrt{2\pi(2n)}(2n)^{2n} e^{-2n}} = \frac{e^n z^{2n+1}}{\sqrt{2} 2^n n^n}. \tag{5.4}$$

All the functions appearing in (5.4) are easily differentiated with respect to n. Before we differentiate, we can simplify by ignoring the constant factor $\sqrt{2}$ and by taking the natural logarithm of both sides. We get

$$\log(t(n)) \approx n + (2n + 1)\log z - n \log 2 - n \log n. \tag{5.5}$$

(Since logarithm is an increasing function, the value of n giving the maximum of (5.5) is the same as that in (5.4).) Differentiating, we get

$$\frac{d(n + (2n + 1)\log z - n \log 2 - n \log n)}{dn} = 1 + 2\log z - \log 2 - \log n - 1. \tag{5.6}$$

This simplifies to $\log(z^2/2) - \log n$, which equals zero when $n = z^2/2$. We have given a heuristic argument that the maximum term of the convergent series occurs near $n = z^2/2$. Thus we also know that the minimum term of the divergent series $td(n, z)$ occurs near $n = z^2/2$.

Heaviside's exponential series

The British electrical engineer Oliver Heaviside (1850–1925) used (without proof) the remarkable series

$$e^x = \sum_{n=-\infty}^{\infty} \frac{x^{n+\gamma}}{\Gamma(n+\gamma+1)}, \tag{6.1}$$

where γ is any complex number, in his work on electromagnetic theory [**7**]. The upper tail of this series is convergent for all x and all γ, while the lower tail is divergent if γ is not an integer. Thus we must view (6.1) and (6.2)–(6.5) as being true in general only in a formal sense. However, if for example $\gamma = 0$, this series becomes

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Here we use $\Gamma(n+1) = n!$ for $n \geq 0$ and we interpret $1\Gamma(n+1)$ when n is a negative integer.

If we set $\gamma = -1/2$ in (6.1) we get

$$e^x = \sum_{n=-\infty}^{\infty} \frac{x^{n-1/2}}{\Gamma(n+1/2)} = \sum_{n=1}^{\infty} \frac{x^{n-1/2}}{\Gamma(n+1/2)} + \sum_{n=0}^{\infty} \frac{x^{-n-1/2}}{\Gamma(-n+1/2)}. \tag{6.2}$$

Now the values of the gamma function become elementary. We have

$$\Gamma(n+1/2) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi}$$

when n is positive or zero and

$$\Gamma(n+1/2) = \frac{(-1)^n 2^n}{1\cdot 3\cdot 5\cdots(2n-1)}\sqrt{\pi}$$

for negative n. Substituting these values for $\Gamma(n+1/2)$ into (6.2) we have

$$e^x = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{2^n x^{n-1/2}}{1\cdot 3\cdot 5\cdots(2n-1)} + \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{2^n x^{n+1/2}}. \tag{6.3}$$

If we put $x = z^2/2$ we get

$$e^{z^2/2} = \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{z^{2n-1}}{1\cdot 3\cdot 5\cdots(2n-1)} + \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{z^{2n+1}}. \tag{6.4}$$

Multiply both sides by $e^{-z^2/2}/2$ to get

$$\frac{1}{2} = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{z^{2n-1}}{1\cdot 3\cdot 5\cdots(2n-1)} + \frac{e^{-z^2/2}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{z^{2n+1}}. \tag{6.5}$$

Notice that the first series in (6.5) is the convergent series we have been studying and the second series is the divergent series. Recalling the derivations of these series by integration by parts, we see that relation (6.5) emerges from the simple relation

$$\frac{1}{2} = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2}\,dt = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-t^2/2}\,dt + \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-t^2/2}\,dt. \tag{6.6}$$

Heaviside's exponential series can be viewed as an example of a generalized Taylor's series using fractional derivatives. This was explored by G. H. Hardy in [**6**], by Raina and Koul in [**14**] and by the first author in [**11**].

Final thoughts

The series (1.1) was first studied by Laplace [**10**] in 1820. Many mathematicians of the eighteenth century used divergent series freely, including Euler. The nineteenth century saw the rise of rigor in analysis, and divergent series were cast out of legitimate mathematics. Yet applied mathematicians continued to need divergent series. Finally Henri Poincaré provided a rigorous theory of asymptotic series [**13**] in 1886. Writing in 1904 James Pierpont [**12**] observed: "It is indeed a strange vicissitude of our science that these series, which early in the century were supposed to be banished once and for all from rigorous mathematics, should at its close be knocking at the door for readmission." Today, nearly one hundred years later, the theory of asymptotic series is a venerable chapter in mathematics. See the book [**8**] by Morris Kline for a good history of divergent series.

There are several good books available for the reader who wishes to pursue this subject. The book by G. H. Hardy [**5**] is a classic. Knopp's book [**9**] has two fine chapters on this subject. Other good books are by de Bruijn [**1**], Copson [**2**], Dingle [**3**], and Erdelyi [**4**].

References

1. N. G. de Bruijn, *Asymptotic Methods in Analysis*, Dover, 1981.
2. E. T. Copson, *Asymptotic Expansions*, Cambridge University Press, 1965.
3. R. B. Dingle, *Asymptotic Expansions*, Academic Press, 1973.
4. A. Erdelyi, *Asymptotic Expansions*, Dover, 1956.
5. G. H. Hardy, *Divergent Series*, Oxford University Press, 1949.
6. G. H. Hardy, Riemann's form of Taylor's series, *J. London Math. Soc.* **20** (1945), 48–57.
7. O. Heaviside, *Electromagnetic Theory*, vol. 2, Chelsea, 1971.
8. M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972, (Ch. 47, 1096–1121).
9. K. Knopp, *Theory and Application of Infinite Series*, Dover, 1990.
10. P. S. Laplace, *Theorie analytique des probabilites*, 3rd ed., 1820, Vol. 1, Part 2, Chap. 1 = *Oeuvres*, 7, pp. 89–110.
11. T. J. Osler, Taylor's series generalized for fractional derivatives and applications, *SIAM J. Math. Anal.* **2** (1971), 37–48.
12. J. Pierpont, The history of mathematics in the nineteenth century, *Bull. Amer. Math. Soc.* **11** (1904), 136–159.
13. H. Poincaré, Sur les integrales irregulieres des equations lineaires, *Acta Mathematica* **8** (1886), 295–344.
14. R. K. Raina and C. L. Koul, On Weyl fractional calculus, *Proc. Amer. Math. Soc.* **73** (1979), 188–192.

